18.440: Lecture 9

Expectations of discrete random variables

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Outline

Defining expectation

Functions of random variables

Motivation

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Functions of random variables

Motivation

Expectation of a discrete random variable

- Recall: a random variable X is a function from the state space to the real numbers.
- Can interpret X as a quantity whose value depends on the outcome of an experiment.
- Say X is a **discrete** random variable if (with probability one) it takes one of a countable set of values.
- For each a in this countable set, write $p(a) := P\{X = a\}$. Call p the **probability mass function**.
- The **expectation** of X, written $E[X]$, is defined by

$$E[X] = \sum_{x:p(x)>0} xp(x).$$

- Represents weighted average of possible values X can take, each value being weighted by its probability.

Simple examples

- Suppose that a random variable X satisfies $P\{X = 1\} = .5$, $P\{X = 2\} = .25$ and $P\{X = 3\} = .25$.
- What is $E[X]$?
- Answer: $.5 \times 1 + .25 \times 2 + .25 \times 3 = 1.75$.
- Suppose $P\{X = 1\} = p$ and $P\{X = 0\} = 1 - p$. Then what is $E[X]$?
- Answer: p.
- Roll a standard six-sided die. What is the expectation of number that comes up?
- Answer: $\frac{1}{6}1 + \frac{1}{6}2 + \frac{1}{6}3 + \frac{1}{6}4 + \frac{1}{6}5 + \frac{1}{6}6 = \frac{21}{6} = 3.5$.

Expectation when state space is countable

- If the state space S is countable, we can give **SUM OVER STATE SPACE** definition of expectation:

$$E[X] = \sum_{s \in S} P\{s\} X(s).$$

- Compare this to the **SUM OVER POSSIBLE X VALUES** definition we gave earlier:

$$E[X] = \sum_{x: p(x) > 0} x p(x).$$

- Example: toss two coins. If X is the number of heads, what is $E[X]$?
- State space is $\{(H,H),(H,T),(T,H),(T,T)\}$ and summing over state space gives $E[X] = \frac{1}{4}2 + \frac{1}{4}1 + \frac{1}{4}1 + \frac{1}{4}0 = 1$.

A technical point

- If the state space S is countable, is it possible that the sum $E[X] = \sum_{s \in S} P(\{s\})X(s)$ somehow depends on the order in which $s \in S$ are enumerated?
- In principle, yes... We only say expectation is defined when $\sum_{s \in S} P(\{x\})|X(s)| < \infty$, in which case it turns out that the sum does not depend on the order.

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Expectation of a function of a random variable

- If X is a random variable and g is a function from the real numbers to the real numbers then $g(X)$ is also a random variable.
- How can we compute $E[g(X)]$?
- Answer:
$$E[g(X)] = \sum_{x:p(x)>0} g(x)p(x).$$
- Suppose that constants a, b, μ are given and that $E[X] = \mu$.
- What is $E[X + b]$?
- How about $E[aX]$?
- Generally, $E[aX + b] = aE[X] + b = a\mu + b$.

More examples

- Let X be the number that comes up when you roll a standard six-sided die. What is $E[X^2]$?
- Let X_j be 1 if the jth coin toss is heads and 0 otherwise. What is the expectation of $X = \sum_{i=1}^n X_j$?
- Can compute this directly as $\sum_{k=0}^n P\{X = k\}k$.
- Alternatively, use symmetry. Expected number of heads should be same as expected number of tails.
- This implies $E[X] = E[n - X]$. Applying $E[aX + b] = aE[X] + b$ formula (with $a = -1$ and $b = n$), we obtain $E[X] = n - E[X]$ and conclude that $E[X] = n/2$.

Additivity of expectation

- If X and Y are distinct random variables, then can one say that $E[X+Y] = E[X] + E[Y]$?
- Yes. In fact, for real constants a and b, we have $E[aX + bY] = aE[X] + bE[Y]$.
- This is called the **linearity of expectation**.
- Another way to state this fact: given sample space S and probability measure P, the expectation $E[\cdot]$ is a **linear** real-valued function on the space of random variables.
- Can extend to more variables $E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$.

More examples

- Now can we compute expected number of people who get own hats in n hat shuffle problem?
- Let X_i be 1 if ith person gets own hat and zero otherwise.
- What is $E[X_i]$, for $i \in \{1, 2, \ldots, n\}$?
- Answer: $1/n$.
- Can write total number with own hat as $X = X_1 + X_2 + \ldots + X_n$.
- Linearity of expectation gives $E[X] = E[X_1] + E[X_2] + \ldots + E[X_n] = n \times 1/n = 1$.

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Why should we care about expectation?

- **Laws of large numbers:** choose lots of independent random variables same probability distribution as X — their average tends to be close to $E[X]$.
- Example: roll $N = 10^6$ dice, let Y be the sum of the numbers that come up. Then Y/N is probably close to 3.5.
- **Economic theory of decision making:** Under "rationality" assumptions, each of us has utility function and tries to optimize its expectation.
- **Financial contract pricing:** under "no arbitrage/interest" assumption, price of derivative equals its expected value in so-called **risk neutral probability**.

Expected utility when outcome only depends on wealth

- Contract one: I'll toss 10 coins, and if they all come up heads (probability about one in a thousand), I'll give you 20 billion dollars.
- Contract two: I'll just give you ten million dollars.
- What are expectations of the two contracts? Which would you prefer?
- Can you find a function $u(x)$ such that given two random wealth variables W_1 and W_2, you prefer W_1 whenever $E[u(W_1)] < E[u(W_2)]$?
- Let's assume $u(0) = 0$ and $u(1) = 1$. Then $u(x) = y$ means that you are indifferent between getting 1 dollar no matter what and getting x dollars with probability $1/y$.

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