

# 18.440: Lecture 36

## Risk Neutral Probability and Black-Scholes

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Risk neutral probability

Black-Scholes

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- ▶ The mathematics of today's lecture will not go far beyond things we know.
- ▶ Main mathematical tasks will be to compute expectations of functions of log-normal random variables (to get the Black-Scholes formula) and differentiate under an integral (to compute risk neutral density functions from option prices).
- ▶ Will spend significant time giving financial *interpretations* of the mathematics.
- ▶ Can interpret this lecture as a sophisticated story problem, illustrating an important application of the probability we have learned in this course (involving probability axioms, expectations, cumulative distribution functions, etc.)

## Risk neutral probability

- ▶ “Risk neutral probability” is a fancy term for “price probability”. (The term “price probability” is arguably more descriptive.)
- ▶ That is, it is a probability measure that you can deduce by looking at prices.
- ▶ For example, suppose somebody is about to shoot a free throw in basketball. What is the price in the sports betting world of a contract that pays one dollar if the shot is made?
- ▶ If the answer is .75 dollars, then we say that the risk neutral probability that the shot will be made is .75.
- ▶ Risk neutral probability is the probability determined by the market betting odds.

- ▶ **Risk neutral probability of event  $A$ :**  $P_{RN}(A)$  denotes

$$\frac{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ if } A \text{ occurs}\}}{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ no matter what}\}}.$$

- ▶ If risk-free interest rate is constant and equal to  $r$  (compounded continuously), then denominator is  $e^{-rT}$ .
- ▶ Assuming no **arbitrage** (i.e., no risk free profit with zero upfront investment),  $P_{RN}$  satisfies axioms of probability. That is,  $0 \leq P_{RN}(A) \leq 1$ , and  $P_{RN}(S) = 1$ , and if events  $A_j$  are disjoint then  $P_{RN}(A_1 \cup A_2 \cup \dots) = P_{RN}(A_1) + P_{RN}(A_2) + \dots$ .
- ▶ **Arbitrage example:** if  $A$  and  $B$  are disjoint and  $P_{RN}(A \cup B) < P_{RN}(A) + P_{RN}(B)$  then we sell contracts paying 1 if  $A$  occurs and 1 if  $B$  occurs, buy contract paying 1 if  $A \cup B$  occurs, pocket difference.

## Risk neutral probability differ vs. “ordinary probability”

- ▶ At first sight, one might think that  $P_{RN}(A)$  describes the market's best guess at the probability that  $A$  will occur.
- ▶ But suppose  $A$  is the event that the government is dissolved and all dollars become worthless. What is  $P_{RN}(A)$ ?
- ▶ Should be 0. Even if people think  $A$  is *likely*, a contract paying a dollar when  $A$  occurs is worthless.
- ▶ Now, suppose there are only 2 outcomes:  $A$  is event that economy booms and everyone prospers and  $B$  is event that economy sags and everyone is needy. Suppose purchasing power of dollar is the same in both scenarios. If people think  $A$  has a .5 chance to occur, do we expect  $P_{RN}(A) > .5$  or  $P_{RN}(A) < .5$ ?
- ▶ Answer:  $P_{RN}(A) < .5$ . People are risk averse. In second scenario they need the money more.

## Non-systemic event

- ▶ Suppose that  $A$  is the event that the Boston Red Sox win the World Series. Would we expect  $P_{RN}(A)$  to represent (a market assessment of) “true probability” in that case?
- ▶ Arguably yes. The amount that *people in general* need or value dollars does not depend much on whether  $A$  occurs (even though the financial needs of specific individuals may depend on heavily on  $A$ ).
- ▶ Even if some people bet based on loyalty, emotion, etc., there will arguably be enough in-it-for-the-money statistical arbitrageurs to keep price near a reasonable guess of “true probability”.

# Extensions of risk neutral probability

- ▶ Definition of risk neutral probability depends on choice of currency (the so-called *numéraire*).
- ▶ Risk neutral probability can be defined for variable times and variable interest rates — e.g., one can take the numéraire to be amount one dollar in a variable-interest-rate money market account has grown to when outcome is known. Can define  $P_{RN}(A)$  to be price of contract paying this amount if and when  $A$  occurs.
- ▶ For simplicity, we focus on fixed time  $T$ , fixed interest rate  $r$  in this lecture.

## Risk neutral probability is objective

- ▶ Check out the contracts on intrade.com.
- ▶ Many financial derivatives are essentially bets of this form.
- ▶ Unlike “true probability” (what does that mean?) the “risk neutral probability” is an objectively measurable price.
- ▶ Pundit: The market predictions are ridiculous. I can estimate probabilities much better than they can.
- ▶ Listener: Then why not make some bets and get rich? If your estimates are so much better, law of large numbers says you'll surely come out way ahead eventually.
- ▶ Pundit: Well, you know... been busy... scruples about gambling... more to life than money...
- ▶ Listener: Yeah, that's what I thought.

## Prices as expectations

- ▶ By assumption, the price of a contract that pays one dollar at time  $T$  if  $A$  occurs is  $P_{RN}(A)e^{-rT}$ .
- ▶ If  $A$  and  $B$  are disjoint, what is the price of a contract that pays 2 dollars if  $A$  occurs, 3 if  $B$  occurs, 0 otherwise?
- ▶ Answer:  $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$ .
- ▶ Generally, in absence of arbitrage, price of contract that pays  $X$  at time  $T$  should be  $E_{RN}(X)e^{-rT}$  where  $E_{RN}$  denotes expectation with respect to the risk neutral probability.
- ▶ Example: if a non-divided paying stock will be worth  $X$  at time  $T$ , then its price today should be  $E_{RN}(X)e^{-rT}$ .
- ▶ (**Aside: So-called fundamental theorem of asset pricing** states that interest-discounted asset prices are martingales with respect to risk neutral probability. Current price of stock being  $E_{RN}[Xe^{-rT}]$  follows from this.)

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# Black-Scholes: main assumption and conclusion

- ▶ More famous MIT professors: Black, Scholes, Merton.
- ▶ 1997 Nobel Prize.
- ▶ **Assumption:** the log of an asset price  $X$  at fixed future time  $T$  is a normal random variable (call it  $N$ ) with some known variance (call it  $T\sigma^2$ ) and some mean (call it  $\mu$ ) with respect to risk neutral probability.
- ▶ **Observation:**  $N$  normal  $(\mu, T\sigma^2)$  implies  $E[e^N] = e^{\mu + T\sigma^2/2}$ .
- ▶ **Observation:** If  $X_0$  is the current price then  $X_0 = E_{RN}[X]e^{-rT} = E_{RN}[e^N]e^{-rT} = e^{\mu + (\sigma^2/2 - r)T}$ .
- ▶ **Observation:** This implies  $\mu = \log X_0 + (r - \sigma^2/2)T$ .
- ▶ **Conclusion:** If  $g$  is any function then the price of a contract that pays  $g(X)$  at time  $T$  is

$$E_{RN}[g(X)]e^{-rT} = E_{RN}[g(e^N)]e^{-rT}$$

where  $N$  is normal with mean  $\mu$  and variance  $T\sigma^2$ .

# Black-Scholes example: European call option

- ▶ A **European call option** on a stock at **maturity date**  $T$ , **strike price**  $K$ , gives the holder the right (but not obligation) to purchase a share of stock for  $K$  dollars at time  $T$ .

The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars.  $SS$

- ▶ If  $X$  is the value of the stock at  $T$ , then the value of the option at time  $T$  is given by  $g(X) = \max\{0, X - K\}$ .
- ▶ **Black-Scholes:** price of contract paying  $g(X)$  at time  $T$  is  $E_{RN}[g(X)]e^{-rT} = E_{RN}[g(e^N)]e^{-rT}$  where  $N$  is normal with variance  $T\sigma^2$ , mean  $\mu = \log X_0 + (r - \sigma^2/2)T$ .
- ▶ Write this as

$$\begin{aligned} e^{-rT} E_{RN}[\max\{0, e^N - K\}] &= e^{-rT} E_{RN}[(e^N - K)1_{N \geq \log K}] \\ &= \frac{e^{-rT}}{\sigma\sqrt{2\pi T}} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}} (e^x - K) dx. \end{aligned}$$

# The famous formula

- ▶ Let  $T$  be time to maturity,  $X_0$  current price of underlying asset,  $K$  strike price,  $r$  risk free interest rate,  $\sigma$  the volatility.
- ▶ We need to compute  $e^{-rT} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}} (e^x - K) dx$  where  $\mu = rT + \log X_0 - T\sigma^2/2$ .
- ▶ Can use complete-the-square tricks to compute the two terms explicitly in terms of standard normal cumulative distribution function  $\Phi$ .
- ▶ Price of European call is  $\Phi(d_1)X_0 - \Phi(d_2)Ke^{-rT}$  where  $d_1 = \frac{\ln(\frac{X_0}{K}) + (r + \frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$  and  $d_2 = \frac{\ln(\frac{X_0}{K}) + (r - \frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$ .

## Determining risk neutral probability from call quotes

- ▶ If  $C(K)$  is price of European call with strike price  $K$  and  $f = f_X$  is risk neutral probability density function for  $X$  at time  $T$ , then  $C(K) = e^{-rT} \int_{-\infty}^{\infty} f(x) \max\{0, x - K\} dx$ .
- ▶ Differentiating under the integral, we find that

$$e^{rT} C'(K) = \int f(x)(-1_{x>K}) dx = -P_{RN}\{X > K\} = F_X(K) - 1,$$

$$e^{rT} C''(K) = f(K).$$

- ▶ We can look up  $C(K)$  for a given stock symbol (say GOOG) and expiration time  $T$  at [cboe.com](http://cboe.com) and work out approximately what  $F_X$  and hence  $f_X$  must be.

## Perspective: implied volatility

- ▶ Risk neutral probability densities derived from call quotes are not quite lognormal in practice. Tails are too fat. Main Black-Scholes assumption is only approximately correct.
- ▶ “Implied volatility” is the value of  $\sigma$  that (when plugged into Black-Scholes formula along with known parameters) predicts the current market price.
- ▶ If Black-Scholes were completely correct, then given a stock and an expiration date, the implied volatility would be the same for all strike prices. In practice, when the implied volatility is viewed as a function of strike price (sometimes called the “volatility smile”), it is not constant.

## Perspective: why is Black-Scholes not exactly right?

- ▶ **Main Black-Scholes assumption:** risk neutral probability densities are lognormal.
- ▶ **Heuristic support for this assumption:** If price goes up 1 percent or down 1 percent each day (with no interest) then the risk neutral probability must be .5 for each (independently of previous days). Central limit theorem gives log normality for large  $T$ .
- ▶ **Replicating portfolio point of view:** can transfer money back and forth between the stock and the risk free asset to ensure that wealth at the end equals option payout. Option price is required initial investment, which is risk neutral expectation of payout. “True probabilities” are irrelevant.
- ▶ **Where arguments for assumption break down:** Fluctuation sizes vary from day to day. Prices can have big jumps.
- ▶ **Fixes:** variable volatility, random interest rates, Lévy jumps....

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## 18.440 Probability and Random Variables

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