18.440: Lecture 31

Central limit theorem

Scott Sheffield

MIT

Outline

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Proving the central limit theorem

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Recall: DeMoivre-Laplace limit theorem

- Let X_i be an i.i.d. sequence of random variables. Write $S_n = \sum_{i=1}^n X_n$.
- Suppose each X_i is 1 with probability p and 0 with probability $q = 1 - p$.
- **DeMoivre-Laplace limit theorem:**

$$\lim_{n\to\infty} P\{a \le \frac{S_n - np}{\sqrt{npq}} \le b\} \to \Phi(b) - \Phi(a).$$

- Here $\Phi(b) - \Phi(a) = P\{a \le Z \le b\}$ when Z is a standard normal random variable.
- $\frac{S_n - np}{\sqrt{npq}}$ describes "number of standard deviations that S_n is above or below its mean".
- Question: Does a similar statement hold if the X_i are i.i.d. but have some other probability distribution?
- **Central limit theorem:** Yes, if they have finite variance.

Example

- Say we roll 10^6 ordinary dice independently of each other.
- Let X_i be the number on the ith die. Let $X = \sum_{i=1}^{10^6} X_i$ be the total of the numbers rolled.
- What is $E[X]$?
- What is $\mathrm{Var}[X]$?
- How about $\mathrm{SD}[X]$?
- What is the probability that X is less than a standard deviations above its mean?
- Central limit theorem: should be about $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$.

Example

- Suppose earthquakes in some region are a Poisson point process with rate λ equal to 1 per year.
- Let X be the number of earthquakes that occur over a ten-thousand year period. Should be a Poisson random variable with rate 10000.
- What is $E[X]$?
- What is $\mathrm{Var}[X]$?
- How about $\mathrm{SD}[X]$?
- What is the probability that X is less than a standard deviations above its mean?
- Central limit theorem: should be about $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$.

General statement

- Let X_i be an i.i.d. sequence of random variables with finite mean μ and variance σ^2.
- Write $S_n = \sum_{i=1}^n X_i$. So $E[S_n] = n\mu$ and $\mathrm{Var}[S_n] = n\sigma^2$ and $\mathrm{SD}[S_n] = \sigma\sqrt{n}$.
- Write $B_n = \frac{X_1+X_2+\ldots+X_n-n\mu}{\sigma\sqrt{n}}$. Then B_n is the difference between S_n and its expectation, measured in standard deviation units.
- **Central limit theorem:**

$$\lim_{n\to\infty} P\{a \le B_n \le b\} \to \Phi(b) - \Phi(a).$$

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Recall: characteristic functions

- Let X be a random variable.
- The **characteristic function** of X is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$. Like $M(t)$ except with i thrown in.
- Recall that by definition $e^{it} = \cos(t) + i\sin(t)$.
- Characteristic functions are similar to moment generating functions in some ways.
- For example, $\phi_{X+Y} = \phi_X\phi_Y$, just as $M_{X+Y} = M_X M_Y$, if X and Y are independent.
- And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_X(at)$.
- And if X has an mth moment then $E[X^m] = i^m\phi_X^{(m)}(0)$.
- Characteristic functions are well defined at all t for all random variables X.

Rephrasing the theorem

- Let X be a random variable and X_n a sequence of random variables.
- Say X_n **converge in distribution** or **converge in law** to X if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ at all $x \in \mathbb{R}$ at which F_X is continuous.
- Recall: the weak law of large numbers can be rephrased as the statement that $A_n = \frac{X_1+X_2+\ldots+X_n}{n}$ converges in law to μ (i.e., to the random variable that is equal to μ with probability one) as $n \to \infty$.
- The central limit theorem can be rephrased as the statement that $B_n = \frac{X_1+X_2+\ldots+X_n-n\mu}{\sigma\sqrt{n}}$ converges in law to a standard normal random variable as $n \to \infty$.

Continuity theorems

- **Lévy's continuity theorem (see Wikipedia):** if

$$\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t)$$

  for all t, then X_n converge in law to X.
- By this theorem, we can prove the central limit theorem by showing $\lim_{n\to\infty} \phi_{B_n}(t) = e^{-t^2/2}$ for all t.
- **Moment generating function continuity theorem:** if moment generating functions $M_{X_n}(t)$ are defined for all t and n and $\lim_{n\to\infty} M_{X_n}(t) = M_X(t)$ for all t, then X_n converge in law to X.
- By this theorem, we can prove the central limit theorem by showing $\lim_{n\to\infty} M_{B_n}(t) = e^{t^2/2}$ for all t.

Proof of central limit theorem with moment generating functions

- Write $Y = \frac{X-\mu}{\sigma}$. Then Y has mean zero and variance 1.
- Write $M_Y(t) = E[e^{tY}]$ and $g(t) = \log M_Y(t)$. So $M_Y(t) = e^{g(t)}$.
- We know $g(0) = 0$. Also $M_Y'(0) = E[Y] = 0$ and $M_Y''(0) = E[Y^2] = \mathrm{Var}[Y] = 1$.
- Chain rule: $M_Y'(0) = g'(0)e^{g(0)} = g'(0) = 0$ and $M_Y''(0) = g''(0)e^{g(0)} + g'(0)^2 e^{g(0)} = g''(0) = 1$.
- So g is a nice function with $g(0) = g'(0) = 0$ and $g''(0) = 1$. Taylor expansion: $g(t) = t^2/2 + o(t^2)$ for t near zero.
- Now B_n is $\frac{1}{\sqrt{n}}$ times the sum of n independent copies of Y.
- So $M_{B_n}(t) = \left(M_Y(t/\sqrt{n})\right)^n = e^{ng(\frac{t}{\sqrt{n}})}$.
- But $e^{ng(\frac{t}{\sqrt{n}})} \approx e^{n(\frac{t}{\sqrt{n}})^2/2} = e^{t^2/2}$, in sense that LHS tends to $e^{t^2/2}$ as n tends to infinity.

Proof of central limit theorem with characteristic functions

- Moment generating function proof only applies if the moment generating function of X exists.
- But the proof can be repeated almost verbatim using characteristic functions instead of moment generating functions.
- Then it applies for any X with finite variance.

Almost verbatim: replace $M_Y(t)$ with $\phi_Y(t)$

- Write $\phi_Y(t) = E[e^{itY}]$ and $g(t) = \log \phi_Y(t)$. So $\phi_Y(t) = e^{g(t)}$.
- We know $g(0) = 0$. Also $\phi'_Y(0) = iE[Y] = 0$ and $\phi''_Y(0) = i^2 E[Y^2] = -\mathrm{Var}[Y] = -1$.
- Chain rule: $\phi'_Y(0) = g'(0)e^{g(0)} = g'(0) = 0$ and $\phi''_Y(0) = g''(0)e^{g(0)} + g'(0)^2 e^{g(0)} = g''(0) = -1$.
- So g is a nice function with $g(0) = g'(0) = 0$ and $g''(0) = -1$. Taylor expansion: $g(t) = -t^2/2 + o(t^2)$ for t near zero.
- Now B_n is $\frac{1}{\sqrt{n}}$ times the sum of n independent copies of Y.
- So $\phi_{B_n}(t) = \left(\phi_Y(t/\sqrt{n})\right)^n = e^{ng(\frac{t}{\sqrt{n}})}$.
- But $e^{ng(\frac{t}{\sqrt{n}})} \approx e^{-n(\frac{t}{\sqrt{n}})^2/2} = e^{-t^2/2}$, in sense that LHS tends to $e^{-t^2/2}$ as n tends to infinity.

Perspective

- The central limit theorem is actually fairly robust. Variants of the theorem still apply if you allow the X_i not to be identically distributed, or not to be completely independent.
- We won't formulate these variants precisely in this course.
- But, roughly speaking, if you have a lot of little random terms that are "mostly independent" — and no single term contributes more than a "small fraction" of the total sum — then the total sum should be "approximately" normal.
- Example: if height is determined by lots of little mostly independent factors, then people's heights should be normally distributed.
- Not quite true... certain factors by themselves can cause a person to be a whole lot shorter or taller. Also, individual factors not really independent of each other.
- *Kind of* true for homogenous population, ignoring outliers.

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