18.440: Lecture 3

Sample spaces, events, probability

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Outline

Outline

Formalizing probability

Sample space

DeMorgan's laws

Axioms of probability

What does "I'd say there's a thirty percent chance it will rain tomorrow" mean?

- **Neurological:** When I think "it will rain tomorrow" the "truth-sensing" part of my brain exhibits 30 percent of its maximum electrical activity.
- **Frequentist:** Of the last 1000 days that meteorological measurements looked this way, rain occurred on the subsequent day 300 times.
- **Market preference ("risk neutral probability"):** The market price of a contract that pays 100 if it rains tomorrow agrees with the price of a contract that pays 30 tomorrow no matter what.
- **Personal belief:** If you offered *me* a choice of these contracts, I'd be indifferent. (What if need for money is different in two scenarios. Replace dollars with "units of utility"?)

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Even more fundamental question: defining a set of possible outcomes

- Roll a die n times. Define a **sample space** to be $\{1,2,3,4,5,6\}^n$, i.e., the set of $a_1,\ldots,a_n$ with each $a_j \in \{1,2,3,4,5,6\}$.
- Shuffle a standard deck of cards. Sample space is the set of 52! permutations.
- Will it rain tomorrow? Sample space is $\{R,N\}$, which stand for "rain" and "no rain."
- Randomly throw a dart at a board. Sample space is the set of points on the board.

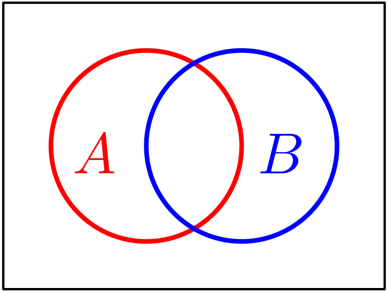
Event: subset of the sample space

- If a set A is comprised of some (but not all) of the elements of B, say A is a **subset** of B and write $A \subset B$.
- Similarly, $B \supset A$ means A is a subset of B (or B is a superset of A).
- If S is a finite sample space with n elements, then there are 2^n subsets of S.
- Denote by $\emptyset$ the set with no elements.

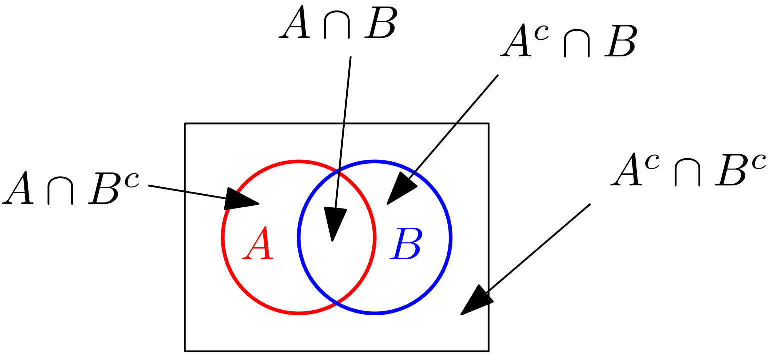
Intersections, unions, complements

- $A \cup B$ means the union of A and B, the set of elements contained in at least one of A and B.
- $A \cap B$ means the intersection of A and B, the set of elements contained on both A and B.
- A^c means complement of A, set of points in whole sample space S but not in A.
- $A \setminus B$ means "A minus B" which means the set of points in A but not in B. In symbols, $A \setminus B = A \cap (B^c)$.
- $\cup$ is associative. So $(A \cup B) \cup C = A \cup (B \cup C)$ and can be written $A \cup B \cup C$.
- $\cap$ is also associative. So $(A \cap B) \cap C = A \cap (B \cap C)$ and can be written $A \cap B \cap C$.

Venn diagrams



Venn diagrams



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- "It will not snow or rain" means "It will not snow and it will not rain."
- If S is event that it snows, R is event that it rains, then $(S \cup R)^c = S^c \cap R^c$
- More generally: $(\cup_{i=1}^n E_i)^c = \cap_{i=1}^n (E_i)^c$
- "It will not both snow and rain" means "Either it will not snow or it will not rain."
- $(S \cap R)^c = S^c \cup R^c$
- $(\cap_{i=1}^n E_i)^c = \cup_{i=1}^n (E_i)^c$

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- $P(A) \in [0,1]$ for all $A \subset S$.
- $P(S) = 1$.
- Finite additivity: $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$.
- Countable additivity: $P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ if $E_i \cap E_j = \emptyset$ for each pair i and j.

- **Neurological:** When I think "it will rain tomorrow" the "truth-sensing" part of my brain exhibits 30 percent of its maximum electrical activity. Should have $P(A) \in [0,1]$ and $P(S) = 1$ but not necessarily $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \emptyset$.
- **Frequentist:** $P(A)$ is the fraction of times A occurred during the previous (large number of) times we ran the experiment. Seems to satisfy axioms...
- **Market preference ("risk neutral probability"):** $P(A)$ is price of contract paying dollar if A occurs divided by price of contract paying dollar regardless. Seems to satisfy axioms, assuming no arbitrage, no bid-ask spread, complete market...
- **Personal belief:** $P(A)$ is amount such that I'd be indifferent between contract paying 1 if A occurs and contract paying $P(A)$ no matter what. Seems to satisfy axioms with some notion of utility units, strong assumption of "rationality"...

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