18.440: Lecture 28

Lectures 17-27 Review

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Outline

Continuous random variables

Problems motivated by coin tossing

Random variable properties

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Random variable properties

Continuous random variables

- Say X is a **continuous random variable** if there exists a **probability density function** $f = f_X$ on $\mathbb{R}$ such that $P\{X \in B\} = \int_B f(x)dx := \int 1_B(x)f(x)dx$.
- We may assume $\int_{\mathbb{R}} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 1$ and f is non-negative.
- Probability of interval $[a, b]$ is given by $\int_a^b f(x)dx$, the area under f between a and b.
- Probability of any single point is zero.
- Define **cumulative distribution function** $F(a) = F_X(a) := P\{X < a\} = P\{X \leq a\} = \int_{-\infty}^{a} f(x)dx$.

Expectations of continuous random variables

- Recall that when X was a discrete random variable, with $p(x) = P\{X = x\}$, we wrote

$$E[X] = \sum_{x:p(x)>0} p(x)x.$$

- How should we define $E[X]$ when X is a continuous random variable?
- Answer: $E[X] = \int_{-\infty}^{\infty} f(x)xdx$.
- Recall that when X was a discrete random variable, with $p(x) = P\{X = x\}$, we wrote

$$E[g(X)] = \sum_{x:p(x)>0} p(x)g(x).$$

- What is the analog when X is a continuous random variable?
- Answer: we will write $E[g(X)] = \int_{-\infty}^{\infty} f(x)g(x)dx$.

Variance of continuous random variables

- Suppose X is a continuous random variable with mean μ.
- We can write $\mathrm{Var}[X] = E[(X-\mu)^2]$, same as in the discrete case.
- Next, if $g = g_1 + g_2$ then $E[g(X)] = \int g_1(x)f(x)dx + \int g_2(x)f(x)dx = \int \big(g_1(x) + g_2(x)\big) f(x)dx = E[g_1(X)] + E[g_2(X)]$.
- Furthermore, $E[ag(X)] = aE[g(X)]$ when a is a constant.
- Just as in the discrete case, we can expand the variance expression as $\mathrm{Var}[X] = E[X^2 - 2\mu X + \mu^2]$ and use additivity of expectation to say that $\mathrm{Var}[X] = E[X^2] - 2\mu E[X] + E[\mu^2] = E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - E[X]^2$.
- This formula is often useful for calculations.

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It's the coins, stupid

- Much of what we have done in this course can be motivated by the i.i.d. sequence X_i where each X_i is 1 with probability p and 0 otherwise. Write $S_n = \sum_{i=1}^n X_n$.
- **Binomial** (S_n — number of heads in n tosses), **geometric** (steps required to obtain one heads), **negative binomial** (steps required to obtain n heads).
- **Standard normal** approximates law of $\frac{S_n - E[S_n]}{\mathrm{SD}(S_n)}$. Here $E[S_n] = np$ and $\mathrm{SD}(S_n) = \sqrt{\mathrm{Var}(S_n)} = \sqrt{npq}$ where $q = 1 - p$.
- **Poisson** is limit of binomial as $n \to \infty$ when $p = \lambda/n$.
- **Poisson point process**: toss one λ/n coin during each length $1/n$ time increment, take $n \to \infty$ limit.
- **Exponential**: time till first event in λ Poisson point process.
- **Gamma distribution**: time till nth event in λ Poisson point process.

Discrete random variable properties derivable from coin toss intuition

- **Sum of two independent binomial random variables** with parameters (n_1, p) and (n_2, p) is itself binomial $(n_1 + n_2, p)$.
- **Sum of n independent geometric random variables** with parameter p is negative binomial with parameter (n, p).
- **Expectation of geometric random variable** with parameter p is $1/p$.
- **Expectation of binomial random variable** with parameters (n, p) is np.
- **Variance of binomial random variable** with parameters (n, p) is $np(1-p) = npq$.

Continuous random variable properties derivable from coin toss intuition

- **Sum of n independent exponential random variables** each with parameter λ is gamma with parameters (n, λ).
- **Memoryless properties:** given that exponential random variable X is greater than $T > 0$, the conditional law of $X - T$ is the same as the original law of X.
- Write $p = \lambda/n$. **Poisson random variable expectation** is $\lim_{n\to\infty} np = \lim_{n\to\infty} n\frac{\lambda}{n} = \lambda$. **Variance** is $\lim_{n\to\infty} np(1-p) = \lim_{n\to\infty} n(1-\lambda/n)\lambda/n = \lambda$.
- **Sum of λ_1 Poisson and independent λ_2 Poisson** is a $\lambda_1 + \lambda_2$ Poisson.
- **Times between successive events** in λ Poisson process are independent exponentials with parameter λ.
- **Minimum of independent exponentials** with parameters λ_1 and λ_2 is itself exponential with parameter $\lambda_1 + \lambda_2$.

DeMoivre-Laplace Limit Theorem

- **DeMoivre-Laplace limit theorem (special case of central limit theorem):**

$$\lim_{n\to\infty} P\{a \le \frac{S_n - np}{\sqrt{npq}} \le b\} \to \Phi(b) - \Phi(a).$$

- This is $\Phi(b) - \Phi(a) = P\{a \le X \le b\}$ when X is a standard normal random variable.

Problems

- Toss a million fair coins. Approximate the probability that I get more than $501,000$ heads.
- Answer: well, $\sqrt{npq} = \sqrt{10^6 \times .5 \times .5} = 500$. So we're asking for probability to be over two SDs above mean. This is approximately $1 - \Phi(2) = \Phi(-2)$.
- Roll 60000 dice. Expect to see 10000 sixes. What's the probability to see more than 9800?
- Here $\sqrt{npq} = \sqrt{60000 \times \frac{1}{6} \times \frac{5}{6}} \approx 91.28$.
- And $200/91.28 \approx 2.19$. Answer is about $1 - \Phi(-2.19)$.

Properties of normal random variables

- Say X is a (standard) **normal random variable** if $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.
- Mean zero and variance one.
- The random variable $Y = \sigma X + \mu$ has variance σ^2 and expectation μ.
- Y is said to be normal with parameters μ and σ^2. Its density function is $f_Y(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$.
- Function $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$ can't be computed explicitly.
- Values: $\Phi(-3) \approx .0013$, $\Phi(-2) \approx .023$ and $\Phi(-1) \approx .159$.
- Rule of thumb: "two thirds of time within one SD of mean, 95 percent of time within 2 SDs of mean."

Properties of exponential random variables

- Say X is an **exponential random variable of parameter** λ when its probability distribution function is $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ (and $f(x) = 0$ if $x < 0$).
- For $a > 0$ have
$$F_X(a) = \int_0^a f(x)dx = \int_0^a \lambda e^{-\lambda x}dx = -e^{-\lambda x}\Big|_0^a = 1 - e^{-\lambda a}.$$
- Thus $P\{X < a\} = 1 - e^{-\lambda a}$ and $P\{X > a\} = e^{-\lambda a}$.
- Formula $P\{X > a\} = e^{-\lambda a}$ is very important in practice.
- Repeated integration by parts gives $E[X^n] = n!/\lambda^n$.
- If $\lambda = 1$, then $E[X^n] = n!$. Value $\Gamma(n) := E[X^{n-1}]$ defined for real $n > 0$ and $\Gamma(n) = (n-1)!$.

Defining Γ distribution

- Say that random variable X has gamma distribution with parameters (α, λ) if $f_X(x) = \begin{cases} \frac{(\lambda x)^{\alpha-1} e^{-\lambda x} \lambda}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$.
- Same as exponential distribution when $\alpha = 1$. Otherwise, multiply by $x^{\alpha-1}$ and divide by $\Gamma(\alpha)$. The fact that $\Gamma(\alpha)$ is what you need to divide by to make the total integral one just follows from the definition of Γ.
- Waiting time interpretation makes sense only for integer α, but distribution is defined for general positive α.

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Properties of uniform random variables

- Suppose X is a random variable with probability density function $f(x) = \begin{cases} \frac{1}{\beta-\alpha} & x \in [\alpha, \beta] \\ 0 & x \notin [\alpha, \beta]. \end{cases}$
- Then $E[X] = \frac{\alpha+\beta}{2}$.
- And $\mathrm{Var}[X] = \mathrm{Var}[(\beta-\alpha)Y + \alpha] = \mathrm{Var}[(\beta-\alpha)Y] = (\beta-\alpha)^2 \mathrm{Var}[Y] = (\beta-\alpha)^2/12$.

Distribution of function of random variable

- Suppose $P\{X \leq a\} = F_X(a)$ is known for all a. Write $Y = X^3$. What is $P\{Y \leq 27\}$?
- Answer: note that $Y \leq 27$ if and only if $X \leq 3$. Hence $P\{Y \leq 27\} = P\{X \leq 3\} = F_X(3)$.
- Generally $F_Y(a) = P\{Y \leq a\} = P\{X \leq a^{1/3}\} = F_X(a^{1/3})$
- This is a general principle. If X is a continuous random variable and g is a strictly increasing function of x and $Y = g(X)$, then $F_Y(a) = F_X(g^{-1}(a))$.

Joint probability mass functions: discrete random variables

- If X and Y assume values in $\{1, 2, \ldots, n\}$ then we can view $A_{i,j} = P\{X = i, Y = j\}$ as the entries of an $n \times n$ matrix.
- Let's say I don't care about Y. I just want to know $P\{X = i\}$. How do I figure that out from the matrix?
- Answer: $P\{X = i\} = \sum_{j=1}^{n} A_{i,j}$.
- Similarly, $P\{Y = j\} = \sum_{i=1}^{n} A_{i,j}$.
- In other words, the probability mass functions for X and Y are the row and columns sums of $A_{i,j}$.
- Given the joint distribution of X and Y, we sometimes call distribution of X (ignoring Y) and distribution of Y (ignoring X) the **marginal** distributions.
- In general, when X and Y are jointly defined discrete random variables, we write $p(x, y) = p_{X,Y}(x, y) = P\{X = x, Y = y\}$.

Joint distribution functions: continuous random variables

- Given random variables X and Y, define $F(a,b) = P\{X \le a, Y \le b\}$.
- The region $\{(x,y) : x \le a, y \le b\}$ is the lower left "quadrant" centered at (a,b).
- Refer to $F_X(a) = P\{X \le a\}$ and $F_Y(b) = P\{Y \le b\}$ as **marginal** cumulative distribution functions.
- Question: if I tell you the two parameter function F, can you use it to determine the marginals F_X and F_Y?
- Answer: Yes. $F_X(a) = \lim_{b\to\infty} F(a,b)$ and $F_Y(b) = \lim_{a\to\infty} F(a,b)$.
- Density: $f(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}F(x,y)$.

Independent random variables

- We say X and Y are independent if for any two (measurable) sets A and B of real numbers we have

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}.$$

- When X and Y are discrete random variables, they are independent if $P\{X = x, Y = y\} = P\{X = x\}P\{Y = y\}$ for all x and y for which $P\{X = x\}$ and $P\{Y = y\}$ are non-zero.
- When X and Y are continuous, they are independent if $f(x, y) = f_X(x)f_Y(y)$.

Summing two random variables

- Say we have independent random variables X and Y and we know their density functions f_X and f_Y.
- Now let's try to find $F_{X+Y}(a) = P\{X + Y \le a\}$.
- This is the integral over $\{(x, y) : x + y \le a\}$ of $f(x, y) = f_X(x)f_Y(y)$. Thus,
- $$P\{X + Y \le a\} = \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f_X(x)f_Y(y)dxdy$$
  $$= \int_{-\infty}^{\infty} F_X(a - y)f_Y(y)dy.$$
- Differentiating both sides gives $f_{X+Y}(a) = \frac{d}{da}\int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy = \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy$.
- Latter formula makes some intuitive sense. We're integrating over the set of x, y pairs that add up to a.

Conditional distributions

- Let's say X and Y have joint probability density function $f(x, y)$.
- We can *define* the conditional probability density of X given that $Y = y$ by $f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$.
- This amounts to restricting $f(x, y)$ to the line corresponding to the given y value (and dividing by the constant that makes the integral along that line equal to 1).

Maxima: pick five job candidates at random, choose best

- Suppose I choose n random variables $X_1, X_2, \ldots, X_n$ uniformly at random on $[0,1]$, independently of each other.
- The n-tuple $(X_1, X_2, \ldots, X_n)$ has a constant density function on the n-dimensional cube $[0,1]^n$.
- What is the probability that the *largest* of the X_i is less than a?
- ANSWER: a^n.
- So if $X = \max\{X_1, \ldots, X_n\}$, then what is the probability density function of X?
- Answer: $F_X(a) = \begin{cases} 0 & a < 0 \\ a^n & a \in [0,1] \\ 1 & a > 1 \end{cases}$. And $f_x(a) = F_X'(a) = na^{n-1}$.

General order statistics

- Consider i.i.d random variables $X_1, X_2, \dots, X_n$ with continuous probability density f.
- Let $Y_1 < Y_2 < Y_3 \dots < Y_n$ be list obtained by *sorting* the X_j.
- In particular, $Y_1 = \min\{X_1, \dots, X_n\}$ and $Y_n = \max\{X_1, \dots, X_n\}$ is the maximum.
- What is the joint probability density of the Y_i?
- Answer: $f(x_1, x_2, \dots, x_n) = n! \prod_{i=1}^n f(x_i)$ if $x_1 < x_2 \dots < x_n$, zero otherwise.
- Let $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ be the permutation such that $X_j = Y_{\sigma(j)}$
- Are σ and the vector $(Y_1, \dots, Y_n)$ independent of each other?
- Yes.

Properties of expectation

- Several properties we derived for discrete expectations continue to hold in the continuum.
- If X is discrete with mass function $p(x)$ then $E[X] = \sum_x p(x)x$.
- Similarly, if X is continuous with density function $f(x)$ then $E[X] = \int f(x)xdx$.
- If X is discrete with mass function $p(x)$ then $E[g(x)] = \sum_x p(x)g(x)$.
- Similarly, X if is continuous with density function $f(x)$ then $E[g(X)] = \int f(x)g(x)dx$.
- If X and Y have joint mass function $p(x, y)$ then $E[g(X, Y)] = \sum_y \sum_x g(x, y)p(x, y)$.
- If X and Y have joint probability density function $f(x, y)$ then $E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f(x, y)dxdy$.

Properties of expectation

- For both discrete and continuous random variables X and Y we have $E[X+Y] = E[X] + E[Y]$.
- In both discrete and continuous settings, $E[aX] = aE[X]$ when a is a constant. And $E[\sum a_i X_i] = \sum a_i E[X_i]$.
- But what about that delightful "area under $1 - F_X$" formula for the expectation?
- When X is non-negative with probability one, do we always have $E[X] = \int_0^\infty P\{X > x\}$, in both discrete and continuous settings?
- Define $g(y)$ so that $1 - F_X(g(y)) = y$. (Draw horizontal line at height y and look where it hits graph of $1 - F_X$.)
- Choose Y uniformly on $[0,1]$ and note that $g(Y)$ has the same probability distribution as X.
- So $E[X] = E[g(Y)] = \int_0^1 g(y)dy$, which is indeed the area under the graph of $1 - F_X$.

A property of independence

- If X and Y are independent then $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$.
- Just write $E[g(X)h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x,y)dxdy$.
- Since $f(x,y) = f_X(x)f_Y(y)$ this factors as $\int_{-\infty}^{\infty} h(y)f_Y(y)dy \int_{-\infty}^{\infty} g(x)f_X(x)dx = E[h(Y)]E[g(X)]$.

Defining covariance and correlation

- Now define covariance of X and Y by $\mathrm{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])$.
- Note: by definition $\mathrm{Var}(X) = \mathrm{Cov}(X, X)$.
- Covariance formula $E[XY] - E[X]E[Y]$, or "expectation of product minus product of expectations" is frequently useful.
- If X and Y are independent then $\mathrm{Cov}(X, Y) = 0$.
- Converse is not true.

Basic covariance facts

- $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$
- $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$
- $\mathrm{Cov}(aX, Y) = a\mathrm{Cov}(X, Y)$.
- $\mathrm{Cov}(X_1 + X_2, Y) = \mathrm{Cov}(X_1, Y) + \mathrm{Cov}(X_2, Y)$.
- **General statement of bilinearity of covariance:**

$$\mathrm{Cov}(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j \mathrm{Cov}(X_i, Y_j).$$

- Special case:

$$\mathrm{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{(i,j): i<j} \mathrm{Cov}(X_i, X_j).$$

Defining correlation

- Again, by definition $\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y]$.
- **Correlation** of X and Y defined by

$$\rho(X, Y) := \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.$$

- Correlation doesn't care what units you use for X and Y. If $a > 0$ and $c > 0$ then $\rho(aX + b, cY + d) = \rho(X, Y)$.
- Satisfies $-1 \leq \rho(X, Y) \leq 1$.
- If a and b are positive constants and $a > 0$ then $\rho(aX + b, X) = 1$.
- If a and b are positive constants and $a < 0$ then $\rho(aX + b, X) = -1$.

Conditional probability distributions

- It all starts with the definition of conditional probability: $P(A|B) = P(AB)/P(B)$.
- If X and Y are jointly discrete random variables, we can use this to define a probability mass function for X *given* $Y = y$.
- That is, we write $p_{X|Y}(x|y) = P\{X = x|Y = y\} = \frac{p(x,y)}{p_Y(y)}$.
- In words: first restrict sample space to pairs (x, y) with given y value. Then divide the original mass function by $p_Y(y)$ to obtain a probability mass function on the restricted space.
- We do something similar when X and Y are continuous random variables. In that case we write $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$.
- Often useful to think of sampling (X, Y) as a two-stage process. First sample Y from its marginal distribution, obtain $Y = y$ for some particular y. Then sample X from its probability distribution *given* $Y = y$.

Conditional expectation

- Now, what do we mean by $E[X|Y=y]$? This should just be the expectation of X in the conditional probability measure for X given that $Y=y$.
- Can write this as $E[X|Y=y] = \sum_x xP\{X=x|Y=y\} = \sum_x xp_{X|Y}(x|y)$.
- Can make sense of this in the continuum setting as well.
- In continuum setting we had $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$. So $E[X|Y=y] = \int_{-\infty}^{\infty} x\frac{f(x,y)}{f_Y(y)}dx$

Conditional expectation as a random variable

- Can think of $E[X|Y]$ as a function of the random variable Y. When $Y = y$ it takes the value $E[X|Y = y]$.
- So $E[X|Y]$ is itself a random variable. It happens to depend only on the value of Y.
- Thinking of $E[X|Y]$ as a random variable, we can ask what *its* expectation is. What is $E[E[X|Y]]$?
- **Very useful fact:** $E[E[X|Y]] = E[X]$.
- In words: what you expect to expect X to be *after learning* Y is same as what you *now* expect X to be.
- Proof in discrete case: $E[X|Y = y] = \sum_x xP\{X = x|Y = y\} = \sum_x x\frac{p(x,y)}{p_Y(y)}$.
- Recall that, in general, $E[g(Y)] = \sum_y p_Y(y)g(y)$.
- $E[E[X|Y = y]] = \sum_y p_Y(y) \sum_x x\frac{p(x,y)}{p_Y(y)} = \sum_x \sum_y p(x,y)x = E[X]$.

Conditional variance

- Definition:
  $\mathrm{Var}(X|Y) = E\big[(X - E[X|Y])^2|Y\big] = E\big[X^2 - E[X|Y]^2|Y\big]$.
- $\mathrm{Var}(X|Y)$ is a random variable that depends on Y. It is the variance of X in the conditional distribution for X given Y.
- Note $E[\mathrm{Var}(X|Y)] = E[E[X^2|Y]] - E[E[X|Y]^2|Y] = E[X^2] - E[E[X|Y]^2]$.
- If we subtract $E[X]^2$ from first term and add equivalent value $E[E[X|Y]]^2$ to the second, RHS becomes $\mathrm{Var}[X] - \mathrm{Var}[E[X|Y]]$, which implies following:
- **Useful fact:** $\mathrm{Var}(X) = \mathrm{Var}(E[X|Y]) + E[\mathrm{Var}(X|Y)]$.
- One can discover X in two stages: first sample Y from marginal and compute $E[X|Y]$, then sample X from distribution given Y value.
- Above fact breaks variance into two parts, corresponding to these two stages.

Example

- Let X be a random variable of variance σ_X^2 and Y an independent random variable of variance σ_Y^2 and write $Z = X + Y$. Assume $E[X] = E[Y] = 0$.
- What are the covariances $\mathrm{Cov}(X, Y)$ and $\mathrm{Cov}(X, Z)$?
- How about the correlation coefficients $\rho(X, Y)$ and $\rho(X, Z)$?
- What is $E[Z|X]$? And how about $\mathrm{Var}(Z|X)$?
- Both of these values are functions of X. Former is just X. Latter happens to be a constant-valued function of X, i.e., happens not to actually depend on X. We have $\mathrm{Var}(Z|X) = \sigma_Y^2$.
- Can we check the formula $\mathrm{Var}(Z) = \mathrm{Var}(E[Z|X]) + E[\mathrm{Var}(Z|X)]$ in this case?

Moment generating functions

- Let X be a random variable and $M(t) = E[e^{tX}]$.
- Then $M'(0) = E[X]$ and $M''(0) = E[X^2]$. Generally, nth derivative of M at zero is $E[X^n]$.
- Let X and Y be independent random variables and $Z = X + Y$.
- Write the moment generating functions as $M_X(t) = E[e^{tX}]$ and $M_Y(t) = E[e^{tY}]$ and $M_Z(t) = E[e^{tZ}]$.
- If you knew M_X and M_Y, could you compute M_Z?
- By independence, $M_Z(t) = E[e^{t(X+Y)}] = E[e^{tX}e^{tY}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t)$ for all t.
- In other words, adding independent random variables corresponds to multiplying moment generating functions.

Moment generating functions for sums of i.i.d. random variables

- We showed that if $Z = X + Y$ and X and Y are independent, then $M_Z(t) = M_X(t)M_Y(t)$
- If $X_1 \ldots X_n$ are i.i.d. copies of X and $Z = X_1 + \ldots + X_n$ then what is M_Z?
- Answer: M_X^n. Follows by repeatedly applying formula above.
- This a big reason for studying moment generating functions. It helps us understand what happens when we sum up a lot of independent copies of the same random variable.
- If $Z = aX$ then $M_Z(t) = E[e^{tZ}] = E[e^{taX}] = M_X(at)$.
- If $Z = X + b$ then $M_Z(t) = E[e^{tZ}] = E[e^{tX+bt}] = e^{bt}M_X(t)$.

Examples

- If X is binomial with parameters (p, n) then $M_X(t) = (pe^t + 1 - p)^n$.
- If X is Poisson with parameter $\lambda > 0$ then $M_X(t) = \exp[\lambda(e^t - 1)]$.
- If X is normal with mean 0, variance 1, then $M_X(t) = e^{t^2/2}$.
- If X is normal with mean μ, variance σ^2, then $M_X(t) = e^{\sigma^2 t^2/2 + \mu t}$.
- If X is exponential with parameter $\lambda > 0$ then $M_X(t) = \frac{\lambda}{\lambda - t}$.

Cauchy distribution

- A standard **Cauchy random variable** is a random real number with probability density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$.
- There is a "spinning flashlight" interpretation. Put a flashlight at $(0,1)$, spin it to a uniformly random angle in $[-\pi/2, \pi/2]$, and consider point X where light beam hits the x-axis.
- $F_X(x) = P\{X \leq x\} = P\{\tan\theta \leq x\} = P\{\theta \leq tan^{-1}x\} = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}x$.
- Find $f_X(x) = \frac{d}{dx}F(x) = \frac{1}{\pi}\frac{1}{1+x^2}$.

Beta distribution

- Two part experiment: first let p be uniform random variable $[0,1]$, then let X be binomial (n,p) (number of heads when we toss n p-coins).
- **Given** that $X = a-1$ and $n - X = b-1$ the conditional law of p is called the β distribution.
- The density function is a constant (that doesn't depend on x) times $x^{a-1}(1-x)^{b-1}$.
- That is $f(x) = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}$ on $[0,1]$, where $B(a,b)$ is constant chosen to make integral one. Can show $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.
- Turns out that $E[X] = \frac{a}{a+b}$ and the mode of X is $\frac{(a-1)}{(a-1)+(b-1)}$.

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