18.440: Lecture 23

Sums of independent random variables

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Summing two random variables

- Say we have independent random variables X and Y and we know their density functions f_X and f_Y.
- Now let's try to find $F_{X+Y}(a) = P\{X + Y \le a\}$.
- This is the integral over $\{(x,y) : x + y \le a\}$ of $f(x,y) = f_X(x)f_Y(y)$. Thus,
- $$P\{X + Y \le a\} = \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f_X(x)f_Y(y)dxdy$$
  $$= \int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy.$$
- Differentiating both sides gives $f_{X+Y}(a) = \frac{d}{da}\int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy = \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy$.
- Latter formula makes some intuitive sense. We're integrating over the set of x, y pairs that add up to a.

Independent identically distributed (i.i.d.)

- The abbreviation i.i.d. means independent identically distributed.
- It is actually one of the most important abbreviations in probability theory.
- Worth memorizing.

Summing i.i.d. uniform random variables

- Suppose that X and Y are i.i.d. and uniform on $[0,1]$. So $f_X = f_Y = 1$ on $[0,1]$.
- What is the probability density function of $X + Y$?
- $f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y) dy = \int_0^1 f_X(a-y)$ which is the length of $[0,1] \cap [a-1,a]$.
- That's a when $a \in [0,1]$ and $2-a$ when $a \in [1,2]$ and 0 otherwise.

Review: summing i.i.d. geometric random variables

- A geometric random variable X with parameter p has $P\{X = k\} = (1-p)^{k-1}p$ for $k \geq 1$.
- Sum Z of n independent copies of X?
- We can interpret Z as time slot where nth head occurs in i.i.d. sequence of p-coin tosses.
- So Z is negative binomial (n, p). So $P\{Z = k\} = \binom{k-1}{n-1}p^{n-1}(1-p)^{k-n}p$.

Summing i.i.d. exponential random variables

- Suppose $X_1, \dots X_n$ are i.i.d. exponential random variables with parameter λ. So $f_{X_i}(x) = \lambda e^{-\lambda x}$ on $[0, \infty)$ for all $1 \le i \le n$.
- What is the law of $Z = \sum_{i=1}^n X_i$?
- We claimed in an earlier lecture that this was a gamma distribution with parameters (λ, n).
- So $f_Z(y) = \frac{\lambda e^{-\lambda y}(\lambda y)^{n-1}}{\Gamma(n)}$.
- We argued this point by taking limits of negative binomial distributions. Can we check it directly?
- By induction, would suffice to show that a gamma $(\lambda, 1)$ plus an independent gamma (λ, n) is a gamma $(\lambda, n+1)$.

Summing independent gamma random variables

- Say X is gamma (λ, s), Y is gamma (λ, t), and X and Y are independent.
- Intuitively, X is amount of time till we see s events, and Y is amount of subsequent time till we see t more events.
- So $f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{s-1}}{\Gamma(s)}$ and $f_Y(y) = \frac{\lambda e^{-\lambda y}(\lambda y)^{t-1}}{\Gamma(t)}$.
- Now $f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y) dy$.
- Up to an a-independent multiplicative constant, this is

$$\int_0^a e^{-\lambda(a-y)}(a-y)^{s-1}e^{-\lambda y}y^{t-1}dy = e^{-\lambda a}\int_0^a (a-y)^{s-1}y^{t-1}dy.$$

- Letting $x = y/a$, this becomes $e^{-\lambda a}a^{s+t-1}\int_0^1 (1-x)^{s-1}x^{t-1}dx$.
- This is (up to multiplicative constant) $e^{-\lambda a}a^{s+t-1}$. Constant must be such that integral from $-\infty$ to ∞ is 1. Conclude that $X+Y$ is gamma $(\lambda, s+t)$.

Summing two normal variables

- X is normal with mean zero, variance σ_1^2, Y is normal with mean zero, variance σ_2^2.
- $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{\frac{-x^2}{2\sigma_1^2}}$ and $f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{\frac{-y^2}{2\sigma_2^2}}$.
- We just need to compute $f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y) dy$.
- We could compute this directly.
- Or we could argue with a multi-dimensional bell curve picture that if X and Y have variance 1 then $f_{\sigma_1 X + \sigma_2 Y}$ is the density of a normal random variable (and note that variances and expectations are additive).
- Or use fact that if $A_i \in \{-1, 1\}$ are i.i.d. coin tosses then $\frac{1}{\sqrt{N}} \sum_{i=1}^{\sigma^2 N} A_i$ is approximately normal with variance σ^2 when N is large.
- Generally: if independent random variables X_j are normal (μ_j, σ_j^2) then $\sum_{j=1}^n X_j$ is normal $(\sum_{j=1}^n \mu_j, \sum_{j=1}^n \sigma_j^2)$.

Other sums

- Sum of an independent binomial (m, p) and binomial (n, p)?
- Yes, binomial $(m+n, p)$. Can be seen from coin toss interpretation.
- Sum of independent Poisson λ_1 and Poisson λ_2?
- Yes, Poisson $\lambda_1 + \lambda_2$. Can be seen from Poisson point process interpretation.

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