18.440: Lecture 21

More continuous random variables

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Outline

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Gamma distribution

Cauchy distribution

Beta distribution

Defining gamma function Γ

- Last time we found that if X is geometric with rate 1 and $n \geq 0$ then $E[X^n] = \int_0^\infty x^n e^{-x} dx = n!$.
- This expectation $E[X^n]$ is actually well defined whenever $n > -1$. Set $\alpha = n + 1$. The following quantity is well defined for any $\alpha > 0$:
  $\Gamma(\alpha) := E[X^{\alpha-1}] = \int_0^\infty x^{\alpha-1} e^{-x} dx = (\alpha - 1)!$.
- So $\Gamma(\alpha)$ extends the function $(\alpha - 1)!$ (as defined for *strictly positive* integers α) to the positive reals.
- Vexing notational issue: why define Γ so that $\Gamma(\alpha) = (\alpha - 1)!$ instead of $\Gamma(\alpha) = \alpha!$?
- At least it's kind of convenient that Γ is defined on $(0, \infty)$ instead of $(-1, \infty)$.

Recall: geometric and negative binomials

- The sum X of n independent geometric random variables of parameter p is negative binomial with parameter (n, p).
- Waiting for the nth heads. What is $P\{X = k\}$?
- Answer: $\binom{k-1}{n-1} p^{n-1}(1-p)^{k-n} p$.
- What's the continuous (Poisson point process) version of "waiting for the nth event"?

Poisson point process limit

- Recall that we can approximate a Poisson process of rate λ by tossing N coins per time unit and taking $p = \lambda/N$.
- Let's fix a rational number x and try to figure out the probability that that the nth coin toss happens at time x (i.e., on exactly xNth trials, assuming xN is an integer).
- Write $p = \lambda/N$ and $k = xN$. (Note $p = \lambda x/k$.)
- For large N, $\binom{k-1}{n-1} p^{n-1}(1-p)^{k-n}p$ is

$$\frac{(k-1)(k-2)\ldots(k-n+1)}{(n-1)!}p^{n-1}(1-p)^{k-n}p$$

$$\approx \frac{k^{n-1}}{(n-1)!}p^{n-1}e^{-x\lambda}p = \frac{1}{N}\Big(\frac{(\lambda x)^{(n-1)}e^{-\lambda x}\lambda}{(n-1)!}\Big).$$

Defining Γ distribution

- The probability from previous side, $\frac{1}{N}\left(\frac{(\lambda x)^{(n-1)}e^{-\lambda x}\lambda}{(n-1)!}\right)$ suggests the form for a continuum random variable.
- Replace n (generally integer valued) with α (which we will eventually allow be to be any real number).
- Say that random variable X has gamma distribution with parameters (α, λ) if $f_X(x) = \begin{cases} \frac{(\lambda x)^{\alpha-1}e^{-\lambda x}\lambda}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$.
- Waiting time interpretation makes sense only for integer α, but distribution is defined for general positive α.

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Cauchy distribution

- A standard **Cauchy random variable** is a random real number with probability density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$.
- There is a "spinning flashlight" interpretation. Put a flashlight at $(0,1)$, spin it to a uniformly random angle in $[-\pi/2, \pi/2]$, and consider point X where light beam hits the x-axis.
- $F_X(x) = P\{X \leq x\} = P\{\tan\theta \leq x\} = P\{\theta \leq tan^{-1}x\} = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}x$.
- Find $f_X(x) = \frac{d}{dx}F(x) = \frac{1}{\pi}\frac{1}{1+x^2}$.

Cauchy distribution: Brownian motion interpretation

- The light beam travels in (randomly directed) straight line. There's a windier random path called Brownian motion.
- If you do a simple random walk on a grid and take the grid size to zero, then you get Brownian motion as a limit.
- We will not give a complete mathematical description of Brownian motion here, just one nice fact.
- FACT: start Brownian motion at point (x, y) in the upper half plane. Probability it hits negative x-axis before positive x-axis is $\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\frac{y}{x}$. Linear function of angle between positive x-axis and line through $(0,0)$ and (x,y).
- Start Brownian motion at $(0,1)$ and let X be the location of the first point on the x-axis it hits. What's $P\{X < a\}$?
- Applying FACT, translation invariance, reflection symmetry: $P\{X < x\} = P\{X > -x\} = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\frac{1}{x}$.
- So X is a standard Cauchy random variable.

Question: what if we start at $(0, 2)$?

- Start at $(0, 2)$. Let Y be first point on x-axis hit by Brownian motion. Again, same probability distribution as point hit by flashlight trajectory.
- Flashlight point of view: Y has the same law as $2X$ where X is standard Cauchy.
- Brownian point of view: Y has same law as $X_1 + X_2$ where X_1 and X_2 are standard Cauchy.
- But wait a minute. $\mathrm{Var}(Y) = 4\mathrm{Var}(X)$ and by independence $\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 2\mathrm{Var}(X_2)$. Can this be right?
- Cauchy distribution doesn't have finite variance or mean.
- Some standard facts we'll learn later in the course (central limit theorem, law of large numbers) don't apply to it.

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Beta distribution: Alice and Bob revisited

- Suppose I have a coin with a heads probability p that I don't know much about.
- What do I mean by not knowing anything? Let's say that I think p is equally likely to be any of the numbers $\{0, .1, .2, .3, .4, \ldots, .9, 1\}$.
- Now imagine a multi-stage experiment where I first choose p and then I toss n coins.
- Given that number h of heads is $a-1$, and $b-1$ tails, what's *conditional* probability p was a certain value x?
- $P\Big(p = x | h = (a-1)\Big) = \frac{\frac{1}{11}\binom{n}{a-1}x^{a-1}(1-x)^{b-1}}{P\{h=(a-1)\}}$ which is $x^{a-1}(1-x)^{b-1}$ times a constant that doesn't depend on x.

Beta distribution

- Suppose I have a coin with a heads probability p that I *really* don't know anything about. Let's say p is uniform on $[0, 1]$.
- Now imagine a multi-stage experiment where I first choose p uniformly from $[0, 1]$ and then I toss n coins.
- If I get, say, $a - 1$ heads and $b - 1$ tails, then what is the *conditional* probability density for p?
- Turns out to be a constant (that doesn't depend on x) times $x^{a-1}(1-x)^{b-1}$.
- $\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}$ on $[0, 1]$, where $B(a, b)$ is constant chosen to make integral one. Can be shown that $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.
- What is $E[X]$?
- Answer: $\frac{a}{a+b}$.

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