18.440: Lecture 12

Poisson random variables

Scott Sheffield

MIT

Outline

Poisson random variable definition

Poisson random variable properties

Poisson random variable problems

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Poisson random variable problems

Poisson random variables: motivating questions

- How many raindrops hit a given square inch of sidewalk during a ten minute period?
- How many people fall down the stairs in a major city on a given day?
- How many plane crashes in a given year?
- How many radioactive particles emitted during a time period in which the expected number emitted is 5?
- How many calls to call center during a given minute?
- How many goals scored during a 90 minute soccer game?
- How many notable gaffes during 90 minute debate?
- **Key idea for all these examples:** Divide time into large number of small increments. Assume that during each increment, there is some small probability of thing happening (independently of other increments).

Remember what e is?

- The number e is defined by $e = \lim_{n \to \infty}(1 + 1/n)^n$.
- It's the amount of money that one dollar grows to over a year when you have an interest rate of 100 percent, continuously compounded.
- Similarly, $e^\lambda = \lim_{n \to \infty}(1 + \lambda/n)^n$.
- It's the amount of money that one dollar grows to over a year when you have an interest rate of 100λ percent, continuously compounded.
- It's also the amount of money that one dollar grows to over λ years when you have an interest rate of 100 percent, continuously compounded.
- Can also change sign: $e^{-\lambda} = \lim_{n \to \infty}(1 - \lambda/n)^n$.

Bernoulli random variable with n large and $np = \lambda$

- Let λ be some moderate-sized number. Say $\lambda = 2$ or $\lambda = 3$. Let n be a huge number, say $n = 10^6$.
- Suppose I have a coin that comes up heads with probability λ/n and I toss it n times.
- How many heads do I expect to see?
- Answer: $np = \lambda$.
- Let k be some moderate sized number (say $k = 4$). What is the probability that I see exactly k heads?
- Binomial formula: $\binom{n}{k} p^k (1-p)^{n-k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} p^k (1-p)^{n-k}$.
- This is approximately $\frac{\lambda^k}{k!}(1-p)^{n-k} \approx \frac{\lambda^k}{k!} e^{-\lambda}$.
- A **Poisson random variable** X with parameter λ satisfies $P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}$ for integer $k \geq 0$.

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Probabilities sum to one

- A **Poisson random variable** X with parameter λ satisfies $p(k) = P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}$ for integer $k \geq 0$.
- How can we show that $\sum_{k=0}^{\infty} p(k) = 1$?
- Use Taylor expansion $e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$.

Expectation

- A **Poisson random variable** X with parameter λ satisfies $P\{X=k\} = \frac{\lambda^k}{k!}e^{-\lambda}$ for integer $k \geq 0$.
- What is $E[X]$?
- We think of a Poisson random variable as being (roughly) a Bernoulli (n,p) random variable with n very large and $p = \lambda/n$.
- This would suggest $E[X] = \lambda$. Can we show this directly from the formula for $P\{X=k\}$?
- By definition of expectation

$$E[X] = \sum_{k=0}^{\infty} P\{X=k\}k = \sum_{k=0}^{\infty} k\frac{\lambda^k}{k!}e^{-\lambda} = \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}e^{-\lambda}.$$

- Setting $j = k-1$, this is $\lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}e^{-\lambda} = \lambda$.

Variance

- Given $P\{X = k\} = \frac{\lambda^k}{k!}e^{-\lambda}$ for integer $k \geq 0$, what is $\mathrm{Var}[X]$?
- Think of X as (roughly) a Bernoulli (n, p) random variable with n very large and $p = \lambda/n$.
- This suggests $\mathrm{Var}[X] \approx npq \approx \lambda$ (since $np \approx \lambda$ and $q = 1 - p \approx 1$). Can we show directly that $\mathrm{Var}[X] = \lambda$?
- Compute

$$E[X^2] = \sum_{k=0}^{\infty} P\{X = k\}k^2 = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!}e^{-\lambda} = \lambda \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!}e^{-\lambda}.$$

- Setting $j = k - 1$, this is

$$\lambda \left(\sum_{j=0}^{\infty} (j+1) \frac{\lambda^j}{j!} e^{-\lambda} \right) = \lambda E[X+1] = \lambda(\lambda+1).$$

- Then $\mathrm{Var}[X] = E[X^2] - E[X]^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$.

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- A country has an average of 2 plane crashes per year.
- How reasonable is it to assume the number of crashes is Poisson with parameter 2?
- Assuming this, what is the probability of exactly 2 crashes? Of zero crashes? Of four crashes?
- A city has an average of five major earthquakes a century. What is the probability that there is an earthquake in a given decade (assuming the number of earthquakes per decade is Poisson)?
- If both candidates average one major gaffe per debate, what is the probably that the first has at least one major gaffe and the second doesn't? (What assumptions are we making?)
- A casino deals one million five-card poker hands per year. Approximate the probability that there are exactly 2 royal flush hands during a given year.

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