18.440: Lecture 11

Binomial random variables and repeated trials

Scott Sheffield

MIT

Outline

Bernoulli random variables

Properties: expectation and variance

More problems

Outline

Bernoulli random variables

Properties: expectation and variance

More problems

Bernoulli random variables

- Toss fair coin n times. (Tosses are independent.) What is the probability of k heads?
- Answer: $\binom{n}{k}/2^n$.
- What if coin has p probability to be heads?
- Answer: $\binom{n}{k}p^k(1-p)^{n-k}$.
- Writing $q = 1 - p$, we can write this as $\binom{n}{k}p^k q^{n-k}$
- Can use binomial theorem to show probabilities sum to one:
- $1 = 1^n = (p+q)^n = \sum_{k=0}^{n} \binom{n}{k}p^k q^{n-k}$.
- Number of heads is **binomial random variable with parameters** (n,p).

Examples

- Toss 6 fair coins. Let X be number of heads you see. Then X is binomial with parameters (n, p) given by $(6, 1/2)$.
- Probability mass function for X can be computed using the 6th row of Pascal's triangle.
- If coin is biased (comes up heads with probability $p \neq 1/2$), we can still use the 6th row of Pascal's triangle, but the probability that $X = i$ gets multiplied by $p^i(1-p)^{n-i}$.

Other examples

- Room contains n people. What is the probability that exactly i of them were born on a Tuesday?
- Answer: use binomial formula $\binom{n}{i} p^i q^{n-i}$ with $p = 1/7$ and $q = 1 - p = 6/7$.
- Let $n = 100$. Compute the probability that nobody was born on a Tuesday.
- What is the probability that exactly 15 people were born on a Tuesday?

Outline

Bernoulli random variables

Properties: expectation and variance

More problems

Outline

Bernoulli random variables

Properties: expectation and variance

More problems

Expectation

- Let X be a binomial random variable with parameters (n, p).
- What is $E[X]$?
- Direct approach: by definition of expectation, $E[X] = \sum_{i=0}^{n} P\{X = i\}i$.
- What happens if we modify the nth row of Pascal's triangle by multiplying the i term by i?
- For example, replace the 5th row $(1, 5, 10, 10, 5, 1)$ by $(0, 5, 20, 30, 20, 5)$. Does this remind us of an earlier row in the triangle?
- Perhaps the prior row $(1, 4, 6, 4, 1)$?

Useful Pascal's triangle identity

- Recall that $\binom{n}{i} = \frac{n \times (n-1) \times \ldots \times (n-i+1)}{i \times (i-1) \times \ldots \times (1)}$. This implies a simple but important identity: $i\binom{n}{i} = n\binom{n-1}{i-1}$.
- Using this identity (and $q = 1 - p$), we can write

$$E[X] = \sum_{i=0}^{n} i \binom{n}{i} p^i q^{n-i} = \sum_{i=1}^{n} n \binom{n-1}{i-1} p^i q^{n-i}.$$

- Rewrite this as $E[X] = np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{(i-1)} q^{(n-1)-(i-1)}$.
- Substitute $j = i - 1$ to get

$$E[X] = np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^{(n-1)-j} = np(p+q)^{n-1} = np.$$

Decomposition approach to computing expectation

- Let X be a binomial random variable with parameters (n, p). Here is another way to compute $E[X]$.
- Think of X as representing number of heads in n tosses of coin that is heads with probability p.
- Write $X = \sum_{j=1}^{n} X_j$, where X_j is 1 if the jth coin is heads, 0 otherwise.
- In other words, X_j is the number of heads (zero or one) on the jth toss.
- Note that $E[X_j] = p \cdot 1 + (1-p) \cdot 0 = p$ for each j.
- Conclude by additivity of expectation that

$$E[X] = \sum_{j=1}^{n} E[X_j] = \sum_{j=1}^{n} p = np.$$

Interesting moment computation

- Let X be binomial (n,p) and fix $k \geq 1$. What is $E[X^k]$?
- Recall identity: $i\binom{n}{i} = n\binom{n-1}{i-1}$.
- Generally, $E[X^k]$ can be written as

$$\sum_{i=0}^{n} i\binom{n}{i}p^i(1-p)^{n-i}i^{k-1}.$$

- Identity gives

$$E[X^k] = np\sum_{i=1}^{n}\binom{n-1}{i-1}p^{i-1}(1-p)^{n-i}i^{k-1} =$$

$$np\sum_{j=0}^{n-1}\binom{n-1}{j}p^j(1-p)^{n-1-j}(j+1)^{k-1}.$$

- Thus $E[X^k] = npE[(Y+1)^{k-1}]$ where Y is binomial with parameters $(n-1,p)$.

Computing the variance

- Let X be binomial (n, p). What is $E[X]$?
- We know $E[X] = np$.
- We computed identity $E[X^k] = npE[(Y+1)^{k-1}]$ where Y is binomial with parameters $(n-1, p)$.
- In particular $E[X^2] = npE[Y+1] = np[(n-1)p+1]$.
- So $\mathrm{Var}[X] = E[X^2] - E[X]^2 = np(n-1)p + np - (np)^2 = np(1-p) = npq$, where $q = 1-p$.
- Commit to memory: variance of binomial (n, p) random variable is npq.
- This is n times the variance you'd get with a single coin. Coincidence?

Compute variance with decomposition trick

- $X = \sum_{j=1}^{n} X_j$, so
  $E[X^2] = E[\sum_{i=1}^{n} X_i \sum_{j=1}^{n} X_j] = \sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i X_j]$
- $E[X_i X_j]$ is p if $i = j$, p^2 otherwise.
- $\sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i X_j]$ has n terms equal to p and $(n-1)n$ terms equal to p^2.
- So $E[X^2] = np + (n-1)np^2 = np + (np)^2 - np^2$.
- Thus
  $\mathrm{Var}[X] = E[X^2] - E[X]^2 = np - np^2 = np(1-p) = npq$.

Outline

Bernoulli random variables

Properties: expectation and variance

More problems

Outline

Bernoulli random variables

Properties: expectation and variance

More problems

More examples

- An airplane seats 200, but the airline has sold 205 tickets. Each person, independently, has a .05 chance of not showing up for the flight. What is the probability that more than 200 people will show up for the flight?
- In a 100 person senate, forty people always vote for the Republicans' position, forty people always for the Democrats' position and 20 people just toss a coin to decide which way to vote. What is the probability that a given vote is tied?
- You invite 50 friends to a party. Each one, independently, has a $1/3$ chance of showing up. That is the probability that more than 25 people will show up?

18.440 Probability and Random Variables
Spring 2014