

**18.440 Midterm 2 Solutions, Fall 2011: 50 minutes, 100 points**

1. (20 points) Suppose that a fair die is rolled 72000 times. Each roll turns up a uniformly random member of the set  $\{1, 2, 3, 4, 5, 6\}$  and the rolls are independent of each other. For each  $j \in \{1, 2, 3, 4, 5, 6\}$  let  $X_j$  be the number of times that the die comes up  $j$ .

- (a) Compute  $E[X_3]$  and  $\text{Var}[X_3]$ . **ANSWER:** Take  $n = 72000$ ,  $p = 1/6$ . Then  $E[X_3] = np = 12000$  and  $\text{Var}[X_3] = np(1 - p) = 10000$ .
- (b) Compute  $\text{Var}[X_1 + X_2]$ . **ANSWER:** This counts the number of times that either a one or a two comes up. Each die roll has a  $1/3$  chance of being a 1 or 2. So  $\text{Var}[X_1 + X_2] = n(1/3)(2/3) = 16000$ .
- (c) Use a normal random variable approximation to estimate the probability that  $X_3 > 12100$ . You may use the function  $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a e^{-x^2/2} dx$  in your answer. **ANSWER:** 12100 is one standard deviation above the mean. Approximate probability is  $1 - \Phi(1)$ .

2. (20 points) Suppose that a fair die is rolled just once. Let  $Y$  be 1 if the die comes up 3 and zero otherwise. Let  $Z$  be 1 if the die comes up 2 and zero otherwise.

- (a) Compute the covariance  $\text{Cov}(Y, Z)$  and the variances  $\text{Var}(Y)$  and  $\text{Var}(Z)$ . **ANSWER:**  
 $\text{Cov}(Y, Z) = E[YZ] - E[Y]E[Z] = 0 - 1/36 = -1/36$  and  
 $\text{Var}(Y) = \text{Var}(Z) = (1/6)(5/6) = 5/36$ .
- (b) Compute the covariance of  $3Y + Z$  and  $Y - 3Z$ . **ANSWER:**  
 $\text{Cov}(3Y + Z, Y - 3Z) = 3\text{Var}(Y) - 8\text{Cov}(Y, Z) - 3\text{Var}(Z) = -8\text{Cov}(Y, Z) = 2/9$ .
- (c) What is the conditional expectation of  $Y$  given that  $Z = 0$ ? **ANSWER:**  $1/5$ .

3. (20 points) At a certain track competition, ten athletes take turns throwing javelins. Let  $X_i$  be the distance that the  $i$ th athlete throws the javelin. Suppose that each  $X_i$  is an exponential random variable with an expectation of 50 meters and that the  $X_i$  are independent of each other.

- (a) What is the probability density function for  $X_1$ ? What is the parameter  $\lambda$  of this exponential random variable? **ANSWER:**  
 $\lambda = 1/50$  and  $f(x) = \lambda e^{-\lambda x}$  if  $x > 0$ , and 0 otherwise.

- (b) Compute the probability that the first athlete throws the javelin more than 50 meters. **ANSWER:**  $e^{-50\lambda} = e^{-1}$ .
- (c) Compute the probability that at least one athlete throws the javelin more than 50 meters. **ANSWER:**  $1 - (1 - e^{-1})^{10}$ .
- (d) Compute  $E[\min\{X_1, X_2, \dots, X_{10}\}]$ , i.e., the expectation of the distance that the last place athlete throws the javelin. **ANSWER:** Minimum of ten independent exponentials of rate  $\lambda = 1/50$  is exponential of rate  $10\lambda = 1/5$ . Expectation is  $1/(10\lambda) = 5$  meters.
4. (20 points) Let  $X$  be a uniformly random variable on  $[0, 5]$ .
- (a) Write the probability density function  $f_X$  and the cumulative distribution function  $F_X$ . **ANSWER:**  $f_X(x) = \begin{cases} 1/5 & 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$ .
- (b) What is the moment generating function  $M_X(t)$ ? **ANSWER:**  $\frac{e^{5t}-1}{5t}$ .
- (c) Suppose that  $Y$  is a random variable for which  $M_Y(0) = 1$  and  $M'_Y(0) = 1$  and  $M''_Y(0) = 2$ . What are  $E[Y]$ ,  $E[Y^2]$  and  $\text{Var}[Y]$ ? **ANSWER:**  $E[Y] = 1$ ,  $E[Y^2] = 2$ , and  $\text{Var}[Y] = 2 - 1^2 = 1$ .
5. (20 points) Suppose that on a certain road, the times at which red cars go by a given spot are given by a Poisson point process with rate  $\lambda = 2/\text{hour}$ . Suppose that the times at which green cars go by are also given by a Poisson point process of rate  $\lambda = 2/\text{hour}$ . Similarly, the times at which blue cars go by are given by a Poisson point process of rate  $\lambda = 2/\text{hour}$ . Suppose that these three Poisson point processes are independent of each other.
- (a) Write down the probability density function for the amount of time until the first red car goes by. **ANSWER:** Write  $\lambda = 2$ . Answer is  $\lambda e^{-\lambda x} = 2e^{-2x}$  if  $x > 0$ , and 0 otherwise.
- (c) Compute the expected amount of time until the first car of *any* of the three colors goes by. **ANSWER:**  $1/6$  hour, or 10 minutes.
- (c) Compute the probability that exactly three red cars go by during the first hour. **ANSWER:**  $e^{-\lambda} \lambda^3 / 3! = 4/(3e^2)$ .
- (d) Compute the expected amount of time until at least one car of each of the three colors has gone by. (Hint: does this remind you of the radioactive decay problem?) **ANSWER:**  $(\frac{1}{6} + \frac{1}{4} + \frac{1}{2}) = \frac{11}{12}$  hours, or 55 minutes.

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