

18.440 Midterm 2 Solutions, Fall 2009

1.

- (a) $P\{X < a\} = a^2$ for $a \in (0, 1)$ and thus $F_X(a) = a^2$. Differentiating gives

$$f_X(a) = \begin{cases} 2a & a \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

- (b) $\mathbb{E}[X] = \int_0^1 f_X(x)xdx = \int_0^1 2x^2dx = 2/3$.

- (c) The pair (X, Y) is uniformly distributed on the triangle $\{(x, y) : x \in [0, 1], y \in [0, 1], x > y\}$. Thus, conditioned on X , Y is uniform on $[0, X]$, so $\mathbb{E}[Y|X] = X/2$.

- (d) $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[A_1A_2] - \mathbb{E}[X](\mathbb{E}[1 - X]) = 1/4 - 2/9 = 1/36$.

2.

- (a) As derived in lecture and on problem sets, the collection of two Poisson processes (the earthquake and the flood processes) may be constructed equivalently by first taking a Poisson point process of rate $2 + 3 = 5$ and then independently declaring each point in the process to be an earthquake with probability $3/5$ (and a flood otherwise). Thus, the number N of earthquakes before the first flood satisfies $P\{N = k\} = (3/5)^k(2/5)$.

- (b) One may recall that the sum of independent rate one exponentials is a gamma distribution with $\alpha = 2$ and $\lambda = 1$: the density is thus

$$f(t) = \begin{cases} e^{-\lambda t}t^{\alpha-1}\lambda^\alpha/\Gamma[\alpha] = e^{-t}t & t \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

This can also be derived directly by letting X and Y denote the two exponential random variables and writing

$$f_{X+Y}(t) = \int_{x=0}^t f_X(x)f_Y(t-x)dx = \int_{x=0}^t e^{-x}e^{-(t-x)}dx = te^{-t}.$$

(c)

$$\begin{aligned}\text{Cov}(\min\{E, F, M\}, M) &= \text{Cov}(\min\{E, F, M\}, \min\{E, F, M\}) \\ &\quad + \text{Cov}(\min\{E, F, M\}, M - \min\{E, F, M\})\end{aligned}$$

The memoryless property of exponentials implies that $\min\{E, F, M\}$ and $M - \min\{E, F, M\}$ are independent, so this becomes $\text{Var}(\min\{E, F, M\})$. This is the variance of an exponential of rate 6, which is $1/36$.

3.

(a) $\mathbb{E}[X^2Y^2] - \mathbb{E}[XY]^2 = \mathbb{E}[X^2]\mathbb{E}[Y^2] - \mathbb{E}[X]^2\mathbb{E}[Y]^2 = \mathbb{E}[X^2]\mathbb{E}[Y^2] = \frac{1}{3}$.

(b) $\mathbb{E}[e^{tX}e^{tY}] = \mathbb{E}[e^{tX}]\mathbb{E}[e^{tY}] = e^{t^2/2} \frac{e^t - 1}{t}$.

4.

(a) $\text{Cov}(X, Y) = \text{Cov}(\sum_{i=1}^{60} X_i, \sum_{j=41}^{100} X_j) = \sum_{i=41}^{60} \text{Cov}(X_i, X_i) = 20$.
Since $\text{Var}(X) = \text{Var}(Y) = 60$, we have $\rho(X, Y) = 20/\sqrt{60 \cdot 60} = 1/3$.

(b) Let $Z = \sum_{i=61}^{100} X_i$. Then X and Z are independent with joint density $f(a, b) = \frac{1}{\sqrt{2\pi}\sqrt{60}}e^{-a^2/120} \frac{1}{\sqrt{2\pi}\sqrt{40}}e^{-b^2/80}$. Conditioning on $X + Z = x$ restricts us to the line $a + b = x$, so the conditional density for X will be (for some constant C)

$$f(a) = Cf(a, x - a) = Ce^{-a^2/120}e^{-(x-a)^2/80},$$

where C is chosen so that $\int_{-\infty}^{\infty} f(a)da = 1$.

5.

(a) If X_i is the multiplicative factor during the i th year, then $\mathbb{E}[X_i] = 2 \cdot .4 + .5 \cdot .6 = 1.1$ and $\mathbb{E}[\prod X_i] = 1.1^{100} \sim 13780.6$.

(b) We need to get at least 50 “up” steps. Mean number is 40, variance is $100(.6 \cdot .4) = 24$. So, very roughly, the chance is $1 - \Phi(10/\sqrt{24}) \sim .96$. Despite high expectation, investment usually loses money.

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