

NAME: _____

Fall 2012 18.440 Final Exam: 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (10 points) Lisa's truck has three states: broken (in Lisa's possession), working (in Lisa's possession), and in the shop. Denote these states B , W , and S .

- (i) Each morning the truck starts out B , it has a $1/2$ chance of staying B and a $1/2$ chance of switching to S by the next morning.
- (ii) Each morning the truck starts out W , it has $9/10$ chance of staying W , and a $1/10$ chance of switching to B by the next morning.
- (iii) Each morning the truck starts out S , it has a $1/2$ chance of staying S and a $1/2$ chance of switching to W by the next morning.

Answer the following

- (a) Write the three-by-three Markov transition matrix for this problem.

- (b) If the truck starts out W on one morning, what is the probability that it will start out B two days later?

- (c) Over the long term, what fraction of mornings does the truck start out in each of the three states, B , S , and W ?

2. (10 points) Suppose that X_1, X_2, X_3, \dots is an infinite sequence of independent random variables which are each equal to 1 with probability $1/2$ and -1 with probability $1/2$. Write $Y_n = \sum_{i=1}^n X_i$. Answer the following:

- (a) What is the probability that Y_n reaches 10 before the first time that it reaches -30 ?

(b) In which of the cases below is the sequence Z_n a martingale? (Just circle the corresponding letters.)

(i) $Z_n = X_n + Y_n$

(ii) $Z_n = \prod_{i=1}^n (2X_i + 1)$

(iii) $Z_n = \prod_{i=1}^n (-X_i + 1)$

(iv) $Z_n = \sum_{i=1}^n Y_i$

(v) $Z_n = \sum_{i=2}^n X_i X_{i-1}$

3. (10 points) Ten people throw their hats into a box and then randomly redistribute the hats among themselves (each person getting one hat, all $10!$ permutations equally likely). Let N be the number of people who get their own hats back. Compute the following:

(a) $E[N^2]$

(b) $P(N = 8)$

4. (10 points) When Harry's cell phone is on, the times when he receives new text messages form a Poisson process with parameter $\lambda_T = 3/\text{minute}$. The times at which he receives new email messages form an independent Poisson process with parameter $\lambda_E = 1/\text{minute}$. He receives personal messages on Facebook as an independent Poisson process with rate $\lambda_F = 2/\text{minute}$.

(a) After catching up on existing messages one morning, Harry begins to wait for new messages to arrive. Let X be the amount of time (in minutes) that Harry has to wait to receive his first text message. Write down the probability density function for X .

(b) Compute the probability that Harry receives 10 new messages total (including email, text, and Facebook) during his first two minutes of waiting.

(c) Let Y be the amount of time elapsed before the third email message. Compute $\text{Var}(Y)$.

(d) What is the probability that Harry receives no messages of any kind during his first five minutes of waiting?

5. (10 points) Suppose that X and Y have a joint density function f given by

$$f(x, y) = \begin{cases} 1/\pi & x^2 + y^2 < 1 \\ 0 & x^2 + y^2 \geq 1 \end{cases}.$$

(a) Compute the probability density function f_X for X .

(b) Express $E[\sin(XY)]$ as a double integral. (You don't have to explicitly evaluate the integral.)

6. (10 points) Let X be the number on a standard die roll (i.e., each of $\{1, 2, 3, 4, 5, 6\}$ is equally likely) and Y the number on an independent standard die roll. Write $Z = X + Y$.

(a) Compute the conditional probability $P[X = 6|Z = 8]$.

(b) Compute the conditional expectation $E[Y|Z]$ as a function of Z (for $Z \in \{2, 3, 4, \dots, 12\}$).

7. (10 points) Suppose that X_i are i.i.d. random variables, each of which assumes a value in $\{-1, 0, 1\}$, each with probability $1/3$.

(a) Compute the moment generating function for X_1 .

(b) Compute the moment generating function for the sum $\sum_{i=1}^n X_i$.

8. (10 points) Let X and Y be independent random variables. Suppose X takes values in $\{1, 2\}$ each with probability $1/2$ and Y takes values in $\{1, 2, 3, 4\}$ each with probability $1/4$. Write $Z = X + Y$.

(a) Compute the entropies $H(X)$ and $H(Y)$.

(b) Compute $H(X, Z)$.

(c) Compute $H(X + Y)$.

9. (10 points) Let X be a normal random variable with mean 0 and variance 1.

(a) Compute $\mathbb{E}[e^X]$.

(b) Compute $\mathbb{E}[e^X 1_{X>0}]$.

(c) Compute $\mathbb{E}[X^2 + 2X - 5]$.

10. (10 points) Let X be uniformly distributed random variable on $[0, 1]$.

(a) Compute the variance of X .

(b) Compute the variance of $3X + 5$.

(c) If X_1, \dots, X_n are independent copies of X , and $Z = \max\{X_1, X_2, \dots, X_n\}$, then what is the cumulative distribution function F_Z ?

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