

Fall 2012 18.440 Final Exam: 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (10 points) Lisa's truck has three states: broken (in Lisa's possession), working (in Lisa's possession), and in the shop. Denote these states B, W, and S.
- (i) Each morning the truck starts out B, it has a 1/2 chance of staying B and a 1/2 chance of switching to S by the next morning.
 - (ii) Each morning the truck starts out W, it has 9/10 chance of staying W, and a 1/10 chance of switching to B by the next morning.
 - (iii) Each morning the truck starts out S, it has a 1/2 chance of staying S and a 1/2 chance of switching to W by the next morning.

Answer the following

- (a) Write the three-by-three Markov transition matrix for this problem.

ANSWER: Ordering the states B, W, S , we may write the Markov chain matrix as

$$M = \begin{pmatrix} .5 & 0 & .5 \\ .1 & .9 & 0 \\ 0 & .5 & .5 \end{pmatrix}.$$

- (b) If the truck starts out W on one morning, what is the probability that it will start out B two days later? **ANSWER:** $(9/10)(1/10) + (1/10)(1/2) = .09 + .05 = .14$
- (c) Over the long term, what fraction of mornings does the truck start out in each of the three states, B , S , and W ? **ANSWER:** We find the stationarity probability vector $\pi = (\pi_B, \pi_W, \pi_S) = (1/7, 5/7, 1/7)$ by solving $\pi M = \pi$ (with components of π summing to 1).

2. (10 points) Suppose that X_1, X_2, X_3, \dots is an infinite sequence of independent random variables which are each equal to 1 with probability 1/2 and -1 with probability 1/2. Write $Y_n = \sum_{i=1}^n X_i$. Answer the following:

- (a) What is the probability that Y_n reaches 10 before the first time that it reaches -30 ? **ANSWER:** Probability p satisfies $10p + (-30)(1 - p) = 0$, so $40p = 30$ and $p = 3/4$.

(b) In which of the cases below is the sequence Z_n a martingale? (Just circle the corresponding letters.)

(i) $Z_n = X_n + Y_n$ **ANSWER: NO**

(ii) $Z_n = \prod_{i=1}^n (2X_i + 1)$ **ANSWER: YES**

(iii) $Z_n = \prod_{i=1}^n (-X_i + 1)$ **ANSWER: YES**

(iv) $Z_n = \sum_{i=1}^n Y_i$ **ANSWER: NO**

(v) $Z_n = \sum_{i=2}^n X_i X_{i-1}$ **ANSWER: YES**

3. (10 points) Ten people throw their hats into a box and then randomly redistribute the hats among themselves (each person getting one hat, all $10!$ permutations equally likely). Let N be the number of people who get their own hats back. Compute the following:

(a) $E[N^2]$ **ANSWER:** Let N_i be 1 if i th person gets own hat, zero otherwise. Then

$$E[(\sum N_i)^2] = \sum_{i=1}^{10} \sum_{j=1}^{10} E[N_i N_j] = 90(1/90) + 10(1/10) = 2.$$

(b) $P(N = 8)$ **ANSWER:** There are $\binom{10}{2}$ ways to pick a pair of people to have swapped hats. So answer is $\binom{10}{2}/10!$.

4. (10 points) When Harry's cell phone is on, the times when he receives new text messages form a Poisson process with parameter $\lambda_T = 3/\text{minute}$. The times at which he receives new email messages form an independent Poisson process with parameter $\lambda_E = 1/\text{minute}$. He receives personal messages on Facebook as an independent Poisson process with rate $\lambda_F = 2/\text{minute}$.

(a) After catching up on existing messages one morning, Harry begins to wait for new messages to arrive. Let X be the amount of time (in minutes) that Harry has to wait to receive his first text message. Write down the probability density function for X . **ANSWER:** time is exponential with parameter $\lambda_T = 3$, so density function is $f(x) = 3e^{-3x}$ for $x \geq 0$.

(b) Compute the probability that Harry receives 10 new messages total (including email, text, and Facebook) during his first two minutes of waiting. **ANSWER:** Number total in two minutes is Poisson with rate $\lambda = 2(\lambda_E + \lambda_T + \lambda_F) = 12$. So answer is $\lambda^k e^{-\lambda}/k! = 12^{10} e^{-12}/10!$.

- (c) Let Y be the amount of time elapsed before the third email message. Compute $\text{Var}(Y)$. **ANSWER:** Variance of time till email message is $1/\lambda_E^2 = 1$. Memoryless property and additivity of variance of independent sums gives $\text{Var}(S) = 3$.
- (d) What is the probability that Harry receives no messages of any kind during his first five minutes of waiting? **ANSWER:** Time till first message is exponential with parameter 6. Probability this time exceeds 5 is e^{-30} .

5. (10 points) Suppose that X and Y have a joint density function f given by

$$f(x, y) = \begin{cases} 1/\pi & x^2 + y^2 < 1 \\ 0 & x^2 + y^2 \geq 1 \end{cases}.$$

- (a) Compute the probability density function f_X for X . **ANSWER:**

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \begin{cases} \frac{1}{\pi} 2\sqrt{1-x^2} & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- (b) Express $E[\sin(XY)]$ as a double integral. (You don't have to explicitly evaluate the integral.) **ANSWER:**

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} \sin(xy) dy dx.$$

6. (10 points) Let X be the number on a standard die roll (i.e., each of $\{1, 2, 3, 4, 5, 6\}$ is equally likely) and Y the number on an independent standard die roll. Write $Z = X + Y$.

- (a) Compute the conditional probability $P[X = 6 | Z = 8]$. **ANSWER:** $1/5$
- (b) Compute the conditional expectation $E[Y | Z]$ as a function of Z (for $Z \in \{2, 3, 4, \dots, 12\}$). **ANSWER:**
 $Z = E[Z | Z] = E[X + Y | Z] = E[X | Z] + E[Y | Z]$. By symmetry,
 $E[X | Z] = E[Y | Z] = Z/2$.

7. (10 points) Suppose that X_i are i.i.d. random variables, each of which assumes a value in $\{-1, 0, 1\}$, each with probability $1/3$.

- (a) Compute the moment generating function for X_1 . **ANSWER:**
 $Ee^{tX_1} = (e^{-t} + 1 + e^t)/3$.

(b) Compute the moment generating function for the sum $\sum_{i=1}^n X_i$.

ANSWER: $(e^{-t} + 1 + e^t)^n / 3^n$

8. (10 points) Let X and Y be independent random variables. Suppose X takes values in $\{1, 2\}$ each with probability $1/2$ and Y takes values in $\{1, 2, 3, 4\}$ each with probability $1/4$. Write $Z = X + Y$.

(a) Compute the entropies $H(X)$ and $H(Y)$. **ANSWER:** $\log 2 = 1$ and $\log 4 = 2$.

(b) Compute $H(X, Z)$. **ANSWER:** $\log 2 + \log 4 = \log 8 = 3$.

(c) Compute $H(X + Y)$. **ANSWER:**

$$\sum_{i=1}^6 P(X+Y = i)(-\log P(X+Y = i)) = 2 \cdot \frac{1}{8} \log 8 + 3 \cdot \frac{1}{4} \log 4 = 6/8 + 6/4 = 9/4.$$

9. (10 points) Let X be a normal random variable with mean 0 and variance 1.

(a) Compute $\mathbb{E}[e^X]$. **ANSWER:**

$$\begin{aligned} E(e^X) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^x dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2-2x+1)/2+1/2} dx = \\ &= e^{1/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2} dx = e^{1/2}. \end{aligned}$$

(b) Compute $\mathbb{E}[e^X 1_{X>0}]$. **ANSWER:**

$$\begin{aligned} E(e^X 1_{X>0}) &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^x dx \\ &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2-2x+1)/2+1/2} dx \\ &= e^{1/2} \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2} dx \\ &= e^{1/2} \int_{-1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = e^{1/2}(1 - \Phi(-1)) = e^{1/2}\Phi(1), \end{aligned}$$

where $\Phi(a) = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

(c) Compute $\mathbb{E}[X^2 + 2X - 5]$. **ANSWER:**

$$E[X^2] + 2E[X] - 5 = 1 + 0 - 5 = -4.$$

10. (10 points) Let X be uniformly distributed random variable on $[0, 1]$.

(a) Compute the variance of X . **ANSWER:** $E[X^2] = \int_0^1 x^2 dx = 1/3$
and $E[X] = 1/2$ so $\text{Var}[X] = E[X^2] - E[X]^2 = 1/12$.

(b) Compute the variance of $3X + 5$. **ANSWER:** $9\text{Var}[X] = 3/4$.

(c) If X_1, \dots, X_n are independent copies of X , and
 $Z = \max\{X_1, X_2, \dots, X_n\}$, then what is the cumulative distribution
function F_Z ? **ANSWER:**

$$F_Z(a) = P\{Z \leq a\} = \prod_{i=1}^n P\{X_i \leq a\} = F_{X_1}(a)^n = \begin{cases} 0 & a < 0 \\ a^n & a \in [0, 1] \\ 1 & a > 1 \end{cases}$$

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