18.440 PROBLEM SET 10: DUE MAY 9

A. FROM TEXTBOOK CHAPTER NINE:

1. Problem/Theoretical Exercise 13: Prove that if X can take on any of n possible values with respective probabilities $P_1, \ldots, P_n$, then $H(X)$ is maximized when $P_i = 1/n, i = 1, \ldots, n$. What is $H(X)$ equal to in this case?

2. Problem/Theoretical Exercise 15: A coin having probability $p = 2/3$ of coming up heads is flipped 6 times. Compute the entropy of the outcome of this experiment. (An "outcome" is a full toss sequence, e.g., $\{H, T, T, T, H, H\}$.)

3. Problem/Theoretical Exercise 17: Show that, for any discrete random variable X and function f,

$$H(f(X)) \leq H(X).$$

B. Solve the following problems:

1. Suppose Harriet has 15 dollars. Her plan is to make one dollar bets on fair coin tosses until her wealth reaches either 0 or 50, and then to go home. What is the expected amount of money that Harriet will have when she goes home? What is the probability that she will have 50 when she goes home?

2. Harriet uses the phrase "I think A" in a precise way. It means "the probability of the event A, conditioned on what I know now, is at least .5". Which of the following statements is true:

    (a) Harriet thinks she will lose.
    (b) Harriet thinks that there will be a time when she thinks she will win.
    (c) Harriet thinks that her amount of money will reach 20 and subsequently reach 10 before the game is over.

3. Consider a contract that at time N will be worth either 100 or 0. Let $S(n)$ be its price at time n for $0 \leq n \leq N$. If $S(n)$ is a martingale, and $S(0) = 47$, then what is the probability that the contract will be worth 100 at time N?

C. Complete the derivation of the Black-Scholes formula for European call options, as outlined in the lecture slides, by explicitly computing

$$E[g(e^N)]e^{-rT},$$

where N is a normal random variable with mean $\mu = \log X_0 + (r - \sigma^2/2)T$ and $g(x) = \max\{0, x - K\}$.

D. Recall Pedro's story from lecture: there is a safe asset (whose value does not change) and a risky asset whose value goes up 15 percent with probability .53 and down 15 percent with probability .47 (independently of everything else) each month. But now suppose that Pedro has the option to divide his money between the risk-free and the risky assets.

1. If Pedro wishes to maximize the expected log of his wealth after one month, what fraction of his wealth should he invest in the risky asset?

2. If Pedro is allowed to rebalance his wealth between the two assets every month, what strategy should he use to maximize the expected log of the amount of money he will have after 100 years (1200 months)? (In particular, what fraction of his wealth should he invest in the risky asset each month? Should it be the same fraction every month?)

3. Suppose that Pedro uses this strategy for 100 years. Compute the mean and standard deviation of the log of Pedro's wealth at the end. Pedro's wealth itself after 100 years is a random variable. Estimate its median.

18.440 Probability and Random Variables
Spring 2014