Lecture 14: Cluster States

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A cluster state is a highly entangled rectangular array of qubits. We measure qubits one at a time. The wiring diagram tells us which basis to measure each qubit in, and which order to measure them in. A wiring diagram is represented by connecting up the dots (which represent qubits) with lines. A junction of two separate lines represents a gate. These gates do not have to be unitary, but if done right, are.

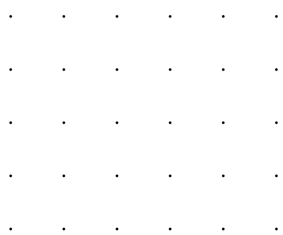

Figure 1: Unconnected cluster states

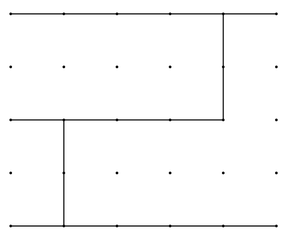

Figure 2: Wiring diagram for two gates

Measurement on a wiring diagram is done by first measuring all the qubits that are not in the wiring diagram (i.e. unconnected) in the σ_z basis. Once those qubits are measured, we measure the qubits in the circuit from left to right in the specified basis.

Cluster state given by eigenvalue equations. The neighborhood of a qubit are the up, down, left, and right qubits.

$$K^{(a)} = \sigma_x^{(a)} \otimes \bigotimes_b \sigma_z^{(b)} | b \in \text{neighborhood}(a) \tag{1}$$

The claim is that $K^{(a)}$ commutes with $K^{(b)}$ when $a \neq b$. To show that this is true, we can look at the following cases:

$$\text{neighborhood}(a) \cap \text{neighborhood}(b) = \emptyset \tag{2}$$

When this is true, then there is absolutely no overlap between $K^{(a)}$ and $K^{(b)}$ and thus the two commute.

$$\text{neighborhood}(a) \not\ni b \tag{3}$$

This means that neighborhoods overlap, but that the qubit b is not in the neighborhood of a in an arrangement such as

$$\begin{array}{cccc} . & a_u & . & . \\ a_l & a & a_r & b \\ . & a_d & . & . \end{array}$$

Figure 3: Neighborhoods overlap

$$\begin{aligned} K^{(a)} &= \sigma_x^{(a)} \otimes \sigma_z^{(a_r)} \otimes \cdots & (4) \\ K^{(b)} &= \sigma_x^{(b)} \otimes \sigma_z^{(a_r)} \otimes \cdots & (5) \\ K^{(a)}K^{(b)} &= \sigma_x^{(a)} \otimes \sigma_z^{(a_r)} \otimes \cdots \otimes \sigma_x^{(b)} \otimes \sigma_z^{(a_r)} \otimes \cdots & (6) \end{aligned}$$

And in the third case, a and b are adjacent to each other.

$$\begin{array}{cccc} . & a_u & . & . \\ a_l & a & b & b_r \\ . & a_d & . & . \end{array}$$

Figure 4: a and b are adjacent

$$K^{(a)}K^{(b)} = \sigma_x^{(a)} \otimes \sigma_z^{(b)} \otimes \cdots \otimes \sigma_x^{(a)}\sigma_z^{(b)} \cdots \quad (7)$$

In all three cases $K^{(a)}$ and $K^{(b)}$ both commute, so the claim holds. This means that $K^{(a)}$ are all simultaneously diagonalizable. Any simultaneous eigenvector of $K^{(a)}, a \in C$ (cluster) is a cluster state. Each $K^{(a)}$ has eigenvalue ± 1, making for 2^n vectors of eigenvalues $\{K_a\}$.

$\left|\phi_{\{\kappa_a\}}\right\rangle_C$ is a cluster state with eigenvalue κ_a on qubit a, $\{\kappa_a\} = \{\pm 1\}$. Thus $\left\langle \phi_{\{\kappa_a\}} | \phi_{\{\kappa_a'\}} \right\rangle_C = 0$ if $\{\kappa_a\} \neq \{\kappa_a^{'}\}$. For example,

$$\begin{aligned}
\kappa_b &= +1 && (8)\\
\kappa_a &= -1 && (9)\\
\left\langle \phi_{\{\kappa_a\}} | \phi_{\{\kappa_a'\}} \right\rangle_C = -\left\langle \phi_{\{\kappa_a\}} | K_b | \phi_{\{\kappa_a'\}} \right\rangle_C &= -\left\langle \phi_{\{\kappa_a\}} | \phi_{\{\kappa_a'\}} \right\rangle_C = 0 && (10)
\end{aligned}$$

If $\{\kappa_a\} = \{\kappa_a^{'}\}$ except for $\kappa_b = -\kappa_b^{'}$, then $\sigma_z^{(b)} \left|\phi_{\{\kappa_a\}}\right\rangle_C = \left|\phi_{\{\kappa_a'\}}\right\rangle_C$

$$\begin{aligned}
K_a \sigma_z^{(b)} \left|\phi_{\{\kappa_a\}}\right\rangle_C &= (-1)^{\delta_{ab}} \sigma_z^{(b)} K_a \left|\phi_{\{\kappa_a\}}\right\rangle_C && (11)\\
&= (-1)^{\delta_{ab}} \sigma_z^{(b)} \kappa_a \left|\phi_{\{\kappa_a\}}\right\rangle_C && (12)\\
&= \kappa_a^{'} \sigma_z^{(b)} \left|\phi_{\{\kappa_a\}}\right\rangle_C && (13)
\end{aligned}$$

Cluster state for $\forall_a \kappa_a = 1$, start in state $|\psi\rangle_C = \bigotimes_a |+\rangle_a$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. We then apply S_{ab} to all neighbors a, b.

$$\begin{aligned}
S_{ab} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} && (14)\\
&= \frac{1}{2}\left(I + \sigma_z^{(a)} + \sigma_z^{(b)} - \sigma_z^{(a)} \otimes \sigma_z^{(b)}\right) && (15)
\end{aligned}$$

Here are a few examples of gates that can be made using wiring diagrams:

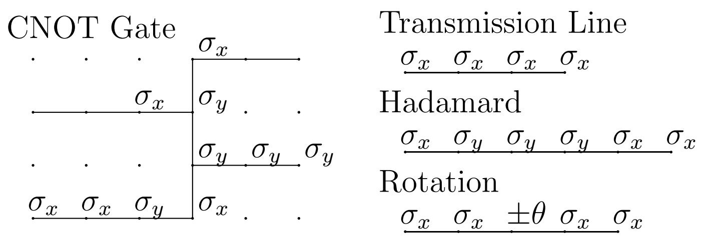

Figure 5: CNOT Gate, Transmission Line, Hadamard, Rotation

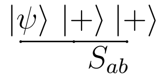

Figure 6: Transmission line

We also know that S_{ab} commutes with $S_{a'b'}$. In the $|0\rangle$, $|1\rangle$ basis, S_{ab} can be represented by a diagonal matrix, which means that they have to commute. $K^{(a)} \bigotimes_{a,b} S_{ab} |+\rangle^{\otimes n}$ is an eigenvector of $K^{(a)}$.

Demonstration of a transmission line effect:

$$S_{ab} |+\rangle |+\rangle = S_{ab} \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \tag{16}$$

$$= \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle - |11\rangle) \tag{17}$$

$$= \frac{1}{\sqrt{2}} (|+\rangle |0\rangle + |-\rangle |1\rangle) \tag{18}$$

With this, we apply S_{ab} and measure both a and b in the $|+\rangle$, $|-\rangle$ basis. This is equivalent to measuring in the $\langle ++| S_{ab}$, $\langle +-| S_{ab}$, $\langle -+| S_{ab}$, $\langle --| S_{ab}$ basis, which is also equivalent to measuring in the $\frac{1}{\sqrt{2}} (\langle 0+| + \langle 1-|)$ basis. In this way we get the teleportation effect on the original $|\psi\rangle$.