

Lecture 2: Basics of Quantum Mechanics

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In this lecture we will cover the basics of Quantum Mechanics which are required to understand the process of quantum computation. To simplify the discussion we assume that all the Hilbert spaces mentioned below are finite-dimensional. The process of quantum computation can be abstracted via the following four postulates.

Postulate 1. *Associated to any isolated physical system is a complex vector space with inner product (that is, a Hilbert space) known as the state space of the system. The state of the system is completely described by a unit vector in this space.*

Qubits were the example of such system that we saw in the previous lecture. In its physical realization as a polarization of a photon we have two basis vectors: $|\uparrow\rangle$ and $|\leftrightarrow\rangle$ representing the vertical and the horizontal polarizations respectively. In this basis vector polarized at angle θ can be expressed as $\cos\theta|\leftrightarrow\rangle - \sin\theta|\uparrow\rangle$.

An important property of a quantum system is that multiplying a quantum state by a unit complex factor ($e^{i\theta}$) yields the same complex state. Therefore $e^{i\theta}|\uparrow\rangle$ and $|\uparrow\rangle$ represent essentially the same state.

Notation 1. *State χ is denoted by $|\chi\rangle$ (often called a ket) is a column vector, e.g.,*

$$\begin{pmatrix} 1/2 \\ i\sqrt{3}/2 \\ 0 \end{pmatrix}$$

$|\chi\rangle^\dagger = \langle\chi|$ (often called a bra) denotes a conjugate transpose of $|\chi\rangle$. In the previous example we would get $(1/2, -i\sqrt{3}/2, 0)$. It is easy to verify that $\langle\chi|\chi\rangle = 1$ and $\langle x|y\rangle \leq 1$.

Postulate 2. *Evolution of a closed quantum system is described by a unitary transformation. If $|\psi\rangle$ is the state at time t , and $|\psi'\rangle$ is the state at time t' , then $|\psi'\rangle = U|\psi\rangle$ for some unitary operator U which depends only on t and t' .*

Definition 1. *A unitary operator is a linear operator that takes unit vectors to unit vectors.*

For every ψ , $\langle\psi|U^\dagger U|\psi\rangle = 1$ and therefore $U^\dagger U = I$. Here by A^\dagger we denote the *adjoint* operator of A , that is, the operator that satisfies $(\langle x|A^\dagger)^\dagger = A|x\rangle$ for every x .

Definition 2. *A Hermitian operator is an operator that satisfies $A^\dagger = A$.*

Commonly used operators on qubits are *Pauli* matrices $I, \sigma_x, \sigma_y, \sigma_z$ and Hadamard transform H described as follows.

$$\begin{aligned} \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{Maps: } |0\rangle \rightarrow |1\rangle; |1\rangle \rightarrow |0\rangle; \frac{|0\rangle+|1\rangle}{\sqrt{2}} \mapsto \frac{|0\rangle-|1\rangle}{\sqrt{2}} \\ \sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \text{Maps: } |0\rangle \rightarrow i|1\rangle; |1\rangle \rightarrow -i|0\rangle \\ \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{Maps: } |0\rangle \rightarrow |0\rangle; |1\rangle \rightarrow -|1\rangle \\ H &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \text{Maps: } |0\rangle \rightarrow \frac{|0\rangle+|1\rangle}{\sqrt{2}}; |1\rangle \rightarrow \frac{|0\rangle-|1\rangle}{\sqrt{2}} \end{aligned}$$

Postulate 2 stems from Schrödinger equation for physical systems, namely

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

where H is a fixed Hermitian operator known as the Hamiltonian of a closed system.

Postulate 3. *Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. These are operators acting on a state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is $|\psi\rangle$ immediately before the measurement then the probability that result m occurs is given by*

$$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle ,$$

and the state of the system after the measurement is

$$\frac{M_m |\psi\rangle}{\sqrt{\langle \psi | M_m^\dagger M_m | \psi \rangle}} .$$

The measurement operators satisfy the completeness equation,

$$\sum_m M_m^\dagger M_m = I .$$

The completeness equation expresses the fact that probabilities sum to one:

$$1 = \sum_m p(m) = \sum_m \langle \psi | M_m^\dagger M_m | \psi \rangle .$$

We will mostly see the following types of measurements. Suppose $|v_1\rangle, |v_2\rangle, \dots, |v_d\rangle$ form an orthonormal basis. Then $\{M_i = |v_i\rangle\langle v_i|\}$ is a quantum measurement. From state $|\psi\rangle$ in this measurement we will obtain

$$\frac{|v_i\rangle\langle v_i|\psi\rangle}{|\langle v_i|\psi\rangle|} \text{ with probability } |\langle v_i|\psi\rangle|^2 .$$

Definition 3. *A projector is a Hermitian matrix with eigenvalues 0 and 1. The subspace with eigenvalue 1 is the subspace associated with this operator.*

Suppose S_1, S_2, \dots, S_k are orthogonal subspaces that span the state space. Then $\{P_i\}$ is a quantum measurement where P_i is the projector onto S_i . We can write

$$|\psi\rangle = \alpha_1 |\psi_1\rangle + \alpha_2 |\psi_2\rangle + \dots + \alpha_k |\psi_k\rangle ,$$

where $|\psi_i\rangle \in S_i$. Then this measurement takes $|\psi\rangle$ to $|\psi_i\rangle$ with probability $|\alpha_i|^2$.

Postulate 4. *The state space of a composite quantum system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n , and system number i is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots \otimes |\psi_n\rangle$.*

Definition 4. *Let S_1 and S_2 be Hilbert spaces with bases $|e_1\rangle, \dots, |e_k\rangle$ and $|f_1\rangle, \dots, |f_l\rangle$ respectively. Then a tensor product of S_1 and S_2 denoted $S_1 \otimes S_2$ is a kl -dimensional space consisting of all the linear combinations of all the possible pairs of original bases elements, that is, of $\{|e_i\rangle \otimes |f_j\rangle\}_{i \leq k, j \leq l}$ ($|v\rangle \otimes |w\rangle$ is often contracted to $|v\rangle |w\rangle$ or $|vw\rangle$).*

In a more concrete matrix representation the tensor product of two vectors is the *Kronecker product* of vectors. For example,

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{3}{5\sqrt{2}} \\ \frac{4}{5\sqrt{2}} \\ -\frac{3}{5\sqrt{2}} \\ -\frac{4}{5\sqrt{2}} \end{pmatrix}$$

The tensor product satisfies the property that the product of two unit vectors is a unit vector. This is to verify as follows. Let $|v_1\rangle = \sum a_i |e_i\rangle$ and $|v_2\rangle = \sum b_j |f_j\rangle$ be two unit vectors. Then

$$|v_1\rangle \otimes |v_2\rangle = \sum a_i |e_i\rangle \otimes \sum b_j |f_j\rangle = \sum a_i b_j |e_i\rangle |f_j\rangle .$$

Therefore,

$$\| |v_1\rangle \otimes |v_2\rangle \|^2 = \sum |a_i b_j|^2 = \sum |a_i|^2 \sum |b_j|^2 = \| |v_1\rangle \|^2 \| |v_2\rangle \|^2$$

Another important property of the tensor product space is that it contains vectors which are not tensor product themselves. For example, it can be easily verified that the vector

$$\frac{1}{\sqrt{2}}(|e_1\rangle |f_2\rangle - |e_2\rangle |f_1\rangle)$$

is not a tensor product itself. Such vectors are called “entangled”.