

Lecture 2

*Lecturer: Daniel A. Spielman***2.1 The Laplacian, again**

We'll begin by redefining the Laplacian. First, let $G_{1,2}$ be the graph on two vertices with one edge. We define

$$L_{G_{1,2}} \stackrel{\text{def}}{=} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Note that

$$x^T L_{G_{1,2}} x = (x_1 - x_2)^2. \quad (2.1)$$

In general, for the graph with n vertices and just one edge between vertices u and v , we can define the Laplacian similarly. For concreteness, I'll call the graph $G_{u,v}$ and define it by

$$L_{G_{u,v}}(i, j) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } i = j \text{ and } i \in u, v \\ -1 & \text{if } i = u \text{ and } j = v, \text{ or } \textit{vice versa}, \\ 0 & \text{otherwise.} \end{cases}$$

For a graph G with edge set E , we now define

$$L(G) \stackrel{\text{def}}{=} \sum_{(u,v) \in E} L(G_{u,v}).$$

Many elementary properties of the Laplacian follow from this definition. In particular, we see that $L_{G_{1,2}}$ has eigenvalues 0 and 2, and so is positive semidefinite, where we recall that a symmetric matrix M is positive semidefinite if all of its eigenvalues are non-negative. Also recall that this is equivalent to

$$x^T M x \geq 0, \text{ for all } x \in \mathbb{R}^n.$$

It follows immediately that the Laplacian of every graph is positive semidefinite. One way to see this is to sum equation (2.1) to get

$$x^T L_G x = \sum_{(u,v) \in E} (x_u - x_v)^2.$$

Remark Since the vertex set really doesn't matter, I actually prefer the notation $L(E)$ where E is a set of edges. Had I used this notation above, it would have eliminated some subscripts. For example, I could have written $L_{\{u,v\}}$ instead of $L_{G_{u,v}}$.

We now sketch the proof of one of the most fundamental facts about Laplacians.

Lemma 2.1.1. *Let $G = (V, E)$ be a connected graph, and let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be the eigenvalues of its Laplacian. Then, $\lambda_2 > 0$.*

Proof. Let x be an eigenvector L_G of eigenvalue 0. Then, we have

$$L_G x = \mathbf{0},$$

and so

$$x^T L_G x = \sum_{(u,v) \in E} (x_u - x_v)^2 = 0.$$

Thus, for each pair of vertices (u, v) connected by an edge, we have $x_u = x_v$. As the graph is connected, we have $x_u = x_v$ for all pairs of vertices u, v , which implies that x is some constant times the all 1s vector. Thus, the eigenspace of 0 has dimension 1. \square

This lemma led Fiedler to consider the magnitude of λ_2 as a measure of how well-connected the graph is. In later lectures, we will see just how good a measure it is. Accordingly, we often call λ_2 the *Fiedler value* of a graph, and v_2 its *Fiedler vector*.

In general, it is pointless to consider unconnected graphs, because their spectra are just the union of the spectra of their components. For example, we have:

Corollary 2.1.2. *Let $G = (V, E)$ be a graph, and let L be its Laplacian. Then, the multiplicity of 0 as an eigenvalue of L equals the number of connected components of G .*

2.2 Some Fundamental Graphs

We now examine the eigenvalues and eigenvectors of the Laplacians of some fundamental graphs. In particular, we will examine

- The complete graph on n vertices, K_n , which has edge set $\{(u, v) : u \neq v\}$.
- The path graph on n vertices, P_n , which has edge set $\{(u, u + 1) : 1 \leq u < n\}$.
- The ring graph on n vertices, R_n , which has all the edges of the path graph, plus the edge $(1, n)$.
- The grid graph on n^2 vertices. To define this graph, we will identify vertices with pairs (u_1, u_2) where $1 \leq u_1 \leq n$ and $1 \leq u_2 \leq n$. Each node (u_1, u_2) has edges to each node that differs by one in just one coordinate.

Lemma 2.2.1. *The Laplacian of K_n has eigenvalue 0 with multiplicity 1 and n with multiplicity $n - 1$.*

Proof. Let L be the Laplacian of K_n . Let x be any vector orthogonal to the all 1s vector. Consider the first coordinate of Lx . It will be $n - 1$ times x_1 , minus

$$\sum_{u>1} x_u = -x_1,$$

as x is orthogonal to $\mathbf{1}$. Thus, x is an eigenvector of eigenvalue n . \square

Lemma 2.2.2. *The Laplacian of R_n has eigenvectors*

$$\begin{aligned} x_k(u) &= \sin(2\pi ku/n), \text{ and} \\ y_k(u) &= \cos(2\pi ku/n), \end{aligned}$$

for $k \leq n/2$. Both of these have eigenvalue $2 - 2\cos(2\pi k/n)$. Note x_0 should be ignored, and y_0 is the all 1s vector. If n is even, then $x_{n/2}$ should also be ignored.

Proof. The best way to see that x_k and y_k are eigenvectors is to plot the graph on the circle using these vectors as coordinates. That they are eigenvectors is geometrically obvious. To compute the eigenvalue, just consider vertex $u = n$. \square

Lemma 2.2.3. *The Laplacian of P_n has the same eigenvalues as R_{2n} , and eigenvectors*

$$x_k(u) = \sin(\pi ku/n + \pi/2n).$$

for $0 \leq k < n$

Proof. This is our first interesting example. We derive the eigenvectors and eigenvalues by treating P_n as a quotient of R_{2n} : we will identify vertex u of P_n with both vertices u and $2n - u$ of R_{2n} . Let z be an eigenvector of R_{2n} in which $z(u) = z(2n - u)$ for all u . I then claim that the first n components of z give an eigenvector of P_n . To see why this is true, note that an eigenvector z of P_n must satisfy for some λ

$$\begin{aligned} 2z(u) - z(u-1) - z(u+1) &= \lambda z(u), \text{ for } 1 < u < n, \text{ and} \\ z(1) - z(2) &= \lambda z(2) \\ z(n) - z(n-1) &= \lambda z(n). \end{aligned}$$

One can now immediately verify that the first n components of such a z satisfy these conditions. The only tricky parts are at the endpoints.

Now, to obtain such a vector z , we take

$$z_k(u) = \sin(2\pi ku/(2n) + \pi/(2n)),$$

which is in the span of x_k and y_k . \square

The quotient construction used in this proof is an example of a generally applicable technique.

2.2.1 Products of Graphs

We will now determine the eigenvalues and eigenvectors of grid graphs. We will go through a more general theory in which we relate the spectra of products graphs to the spectra of the graphs of which they are products.

Definition 2.2.4. Let $G = (V, E)$ and $H = (W, F)$ be graphs. Then $G \times H$ is the graph with vertex set $V \times W$ and edge set

$$\begin{aligned} & \left((v_1, w), (v_2, w) \right) \text{ where } (v_1, v_2) \in E \text{ and} \\ & \left((v, w_1), (v, w_2) \right) \text{ where } (w_1, w_2) \in F. \end{aligned}$$

Theorem 2.2.5. Let $G = (V, E)$ and $H = (W, F)$ be graphs with Laplacian eigenvalues $\lambda_1, \dots, \lambda_n$ and μ_1, \dots, μ_m , and eigenvectors x_1, \dots, x_n and y_1, \dots, y_m , respectively. Then, for each $1 \leq i \leq n$ and $1 \leq j \leq m$, $G \times H$ has an eigenvector z of eigenvalue $\lambda_i + \mu_j$ such that

$$z(v, w) = x_i(v)y_j(w).$$

Proof. To see that this eigenvector has the proper eigenvalue, let L denote the Laplacian of $G \times H$, d_v the degree of node v in G , and e_w the degree of node w in H . We can then verify that

$$\begin{aligned} (Lz)(v, w) &= (d_v + e_w)(x_i(v)y_j(w)) - \sum_{(v, v_2) \in E} (x_i(v_2)y_j(w)) - \sum_{(w, w_2) \in F} (x_i(v)y_j(w_2)) \\ &= (d_v)(x_i(v)y_j(w)) - \sum_{(v, v_2) \in E} (x_i(v_2)y_j(w)) + (e_w)(x_i(v)y_j(w)) - \sum_{(w, w_2) \in F} (x_i(v)y_j(w_2)) \\ &= y_j(w) \left(d_v x_i(v) - \sum_{(v, v_2) \in E} (x_i(v_2)) \right) + x_i(v) \left(e_w y_j(w) - \sum_{(w, w_2) \in F} (x_i(v)y_j(w_2)) \right) \\ &= y_j(w) \lambda_i x_i(v) + x_i(v) \mu_j y_j(w) \\ &= (\lambda_i + \mu_j)(x_i(v)y_j(w)). \end{aligned}$$

□

Thus, the eigenvalues and eigenvectors of the grid graph are completely determined by the eigenvalues and eigenvectors of the path graph. This completely explains the nice spectral embedding of the n -by- n grid graph: its smallest non-zero eigenvalue has multiplicity two and its eigenspace is spanned by the vectors $\mathbf{1} \times v_2$ and $v_2 \times \mathbf{1}$.

2.3 Bounding Eigenvalues

It is rare to be able to precisely determine the eigenvalues of a graph by purely combinatorial arguments. Usually, such arguments only provide bounds on the eigenvalues. The most commonly used tool for bounding eigenvalues is the Courant-Fischer characterization of eigenvalues:

Theorem 2.3.1 (Courant-Fischer). Let A be an n -by- n symmetric matrix and let $1 \leq k \leq n$. Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be the eigenvalues of A . Then,

$$\lambda_k = \min_{S \text{ of dim } k} \max_{x \in S} \frac{x^T A x}{x^T x}, \quad (2.2)$$

and

$$\lambda_k = \max_{S \text{ of dim } n-k-1} \min_{x \in S} \frac{x^T A x}{x^T x}, \quad (2.3)$$

For example, this allows us to obtain an easy upper bound on $\lambda_2(P_n)$. Setting $S = \{x : x \perp \mathbf{1}\}$, and applying (2.3), we obtain

$$\lambda_2(L) \leq \min_{x \perp \mathbf{1}} \frac{x^T A x}{x^T x}.$$

This allows us to obtain an easy upper bound on $\lambda_2(P_n)$. Consider the vector x such that $x(u) = (n+1) - 2u$. This vector satisfies $x \perp \mathbf{1}$, so

$$\begin{aligned} \lambda_2(P_n) &\leq \frac{\sum_{1 \leq u < n} (x(u) - x(u+1))^2}{\sum_u x(u)^2} \\ &= \frac{\sum_{1 \leq u < n} 2^2}{\sum_i (n+1-2u)^2} \\ &\sim \frac{4n}{n(n^2/3)} \\ &= \frac{12}{n^2}. \end{aligned}$$

So, we can easily obtain an upper bound on $\lambda(P_n)$ that is of the right order of magnitude.

It is more difficult to obtain lower bounds. In the next lecture, we will introduce a technique for proving lower bounds.

We will apply the technique to the example of the complete binary tree. The complete binary tree on $n = 2^d - 1$ nodes, B_n , is the graph with edges of the form $(u, 2u)$ and $(u, 2u+1)$ for $u < n/2$. To upper bound $\lambda_2(B_n)$, we construct a vector x as follows. We first set $x(1) = 0$, $x(2) = 1$, and $x(3) = -1$. Then, for every vertex u that we can reach from node 2 without going through node 1, we set $x(u) = 1$. For all the other nodes, we set $x(u) = -1$. We then have

$$\frac{\sum_{(u,v) \in B_n} (x_u - x_v)^2}{\sum_u x_u^2} = \frac{(x_1 - x_2)^2 + (x_1 - x_3)^2}{n} = 2/n.$$

For now, we leave with some lower bounds on λ_n .

Lemma 2.3.2. Let $G = (V, E)$ be a graph and let vertex $w \in V$ have degree d . Then,

$$\lambda_{\max}(G) \geq d.$$

Proof. By (2.2),

$$\lambda_{max}(G) \geq \max_x \frac{x^T L(G)x}{x^T x}.$$

Let x be the vector given by

$$x(u) = \begin{cases} 1 & u = w \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$\frac{\sum_{(u,v) \in E} (x_u - x_v)^2}{\sum x_u^2} = \frac{d}{1} = d.$$

□

We can improve this lemma slightly.

Lemma 2.3.3. *Let $G = (V, E)$ be a graph and let vertex $w \in V$ have degree d . Then,*

$$\lambda_{max}(G) \geq d + 1.$$

Proof. Let x be the vector given by

$$x(u) = \begin{cases} d & u = w \\ -1 & \text{if } (u, w) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$\frac{\sum_{(u,v) \in E} (x_u - x_v)^2}{\sum x_u^2} = \frac{d(d+1)^2}{d^2 + d} = d + 1.$$

□

Can we prove a larger lower bound? To find out, let's consider the star graph with n vertices, S_n . It has edges of the form $(1, i)$ for $2 \leq i \leq n$. To determine its eigenvalues, we apply the following lemma

Lemma 2.3.4. *Let $G = (V, E)$ be a graph, and let v_1 and v_2 be vertices of degree one that are both connected to another vertex w . Then, the vector x given by*

$$x(u) = \begin{cases} 1 & u = v_1 \\ -1 & u = v_2 \\ 0 & \text{otherwise,} \end{cases}$$

is an eigenvector of the Laplacian of G of eigenvalue 1.

Proof. One can immediately verify that $Lx = x$.

□

Thus, we see that S_n has eigenvectors of eigenvalue 1 pairing vertices 2 and 3, 3 and 4, up to 2 and n . The space spanned by these vectors has dimension $n - 2$. As the vector $\mathbf{1}$ is also an eigenvector, only one vector may remain. We don't even have to compute it to determine its eigenvalue: the sum of the eigenvalues is the trace of the matrix, which is $2(n - 1)$. The $n - 1$ eigenvectors we have found so far account for $n - 2$ of this sum, so the last eigenvector must have eigenvalue n . This establishes that Lemma 2.3.4 is tight. In fact, this last eigenvector turns out to be the vector constructed in the proof of that lemma.

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