18.409 An Algorithmist's Toolkit	September 15, 2007
Lecture 2	
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1 Administrative Details

- Signup online for scribing.

2 Review of Lecture 1

All of the following are covered in detail in the notes for Lecture 1:

- The definition of L_G, specifically that $L_G = D_G - A_G$, where D_G is a diagonal matrix of degrees and A_G is the adjacency matrix of graph G.
- The action of L_G on a vector x, namely that
$$[L_G x]_i = deg(i)\,(x_i - \text{average of } x \text{ on neighbors of } i)$$
- The eigenvalues of L_G are $\lambda_1 \leq \cdots \leq \lambda_n$ with corresponding eigenvectors $v_1, \ldots, v_n$. The first and the most trivial eigenvector is $v_1 = \mathbf{1}$ with an eigenvalue $\lambda_1 = 0$. We are mostly interested in v_2 and λ_2.

3 Properties of the Laplacian

3.1 Some simple properties

Lemma 1 (Edge Union) *If G and H are two graphs on the same vertex set with **disjoint** edge sets,*

$$L_{G\cup H} = L_G + L_H \text{ (additivity)} \tag{1}$$

Lemma 2 (Isolated Vertices) *If a vertex $i \in G$ is isolated, then the corresponding row and column of the Laplacian are zero, i.e. $[L_G]_{i,j} = [L_G]_{j,i} = 0$ for all j.*

Lemma 3 (Disjoint Union) *These together imply that the Laplacian of the disjoint union of G and H is direct sum of L_G and L_H, i.e.:*

$$L_{G\coprod H} = L_G \oplus L_H = \begin{pmatrix} L_G & 0 \\ 0 & L_H \end{pmatrix} \tag{2}$$

Proof Consider the graph $G \amalg v(H) = (V_G \cup V_H, E_G)$, namely the graph consisting of G along with the vertex set of H as disjoint vertices. Define $v(G) \amalg H$ similarly. By the second remark,

$$L_{G\amalg v(H)} = \begin{pmatrix} L_G & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad L_{v(G)\amalg H} = \begin{pmatrix} 0 & 0 \\ 0 & L_H \end{pmatrix}.$$

By definition, $G \amalg H = (G \amalg v(H)) \cup (v(G) \amalg H)$, and so by the first remark:

$$L_{G\coprod H} = L_G \oplus L_H = \begin{pmatrix} L_G & 0 \\ 0 & L_H \end{pmatrix}. \quad \blacksquare$$

This implies the Laplacian is the direct sum of the Laplacians of the connected components. Thus,

Theorem 4 (Disjoint Union Spectrum) *If L_G has eigenvectors $v_1, \dots, v_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$, and L_H has eigenvectors $w_1, \dots, w_n$ with eigenvalues $\mu_1, \dots, \mu_n$, then $L_{G \amalg H}$ has eigenvectors:*

$$v_1 \oplus \mathbf{0}, \dots, v_n \oplus \mathbf{0}, \mathbf{0} \oplus w_1, \dots, \mathbf{0} \oplus w_n$$

with corresponding eigenvalues:

$$\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n.$$

Proof By the previous lemma,

$$L_{G \coprod H} * (v_1 \oplus \mathbf{0}) = \begin{pmatrix} L_G & 0 \\ 0 & L_H \end{pmatrix} \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 \\ 0 \end{pmatrix}$$

Thus $v_1 \oplus \mathbf{0}$ is an eigenvector of $L_{G \amalg H}$ with eigenvalue λ_1. The rest follow by symmetry. ■

3.2 The Laplacian of an edge

Definition 5 *Let L_e be the Laplacian of the graph on n vertices consisting of just the edge e.*

Example 1 *If e is the edge (v_1, v_2), then*

$$L_e = \begin{pmatrix} 1 & -1 & 0 & & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

By additivity, this lets us write:

$$L_G = \sum_{e \in E} L_e \tag{3}$$

This will allow us to prove a number of facts about the Laplacian by proving them for one edge and adding them up. The more general technique, which we'll use more later, is to bound Laplacians by adding matrices of substructures.

So for an edge e,

$$L_e = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \oplus [\text{zeros}].$$

Note that:

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} = 2 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix},$$

and so $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^T$ is an eigenvector with eigenvalue 2. This decomposition implies:

$$x^T L_e x = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 - x_2)^2. \tag{4}$$

Remark The Laplacian is a quadratic form, specifically:

$$x^T L_G x = x^T (\sum_{e \in E} L_e) x = \sum_{e \in E} x^T L_e x = \sum_{(i,j) \in E} (x_i - x_j)^2 \tag{5}$$

This implies that L is *positive semidefinite.*

3.3 Review of Positive Semidefiniteness

Definition 6 *A symmetric matrix M is* positive semidefinite (PSD) *if* $\forall x \in \mathbb{R}^n$,

$$x^T M x \geq 0.$$

M is positive definite (PD) *if the inequality is strict* $\forall x \neq 0$.

Lemma 7 *M is PSD iff all eigenvalues $\lambda_i \geq 0$. Similarly M is PD iff all eigenvalues $\lambda_i > 0$.*

Proof Let's consider the matrix M in its eigenbasis, that is $M = Q^T \Lambda Q$. Clearly, $y^T \Lambda y = \sum_i \lambda_i y_i^2 \geq 0$ for all $y \in \mathbb{R}^n$ iff $\lambda_i \geq 0$ for all i. Similar for PD matrix. ■

Lemma 8 (PSD Matrix Decomposition) *M is PSD iff there exists a matrix A such that*

$$M = A^T A. \tag{6}$$

Note that A can be $(n \times k)$ for any k, and that it need not be square. Importantly, note that A is not unique.

Proof

($\Rightarrow$) If M is positive semidefinite, recall that M can be diagonalized as

$$M = Q^T \Lambda Q,$$

thus

$$M = Q^T \Lambda^{1/2} \Lambda^{1/2} Q = \left(\Lambda^{1/2} Q\right)^T \left(\Lambda^{1/2} Q\right),$$

where $\Lambda^{1/2}$ has $\sqrt{\lambda_i}$ on the diagonal.

($\Leftarrow$) If $M = A^T A$, then

$$x^T M x = x^T A^T A x = (Ax)^T (Ax)$$

Letting $y = (Ax) \in \mathbb{R}^k$, we see that:

$$x^T M x = y^T y = ||y||^2 \geq 0. \quad ■$$

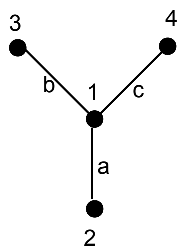
3.4 Factoring the Laplacian

We know from the previous section that we can factor L as $A^T A$ using eigenvectors, but there also exists a much nicer factorization which we will show here.

Definition 9 *Let m be the number of edges and n be the number of vertices. Then the* incidence matrix *$\nabla = \nabla_G$ is the $m \times n$ matrix given by:*

$$\nabla_{e,v} = \begin{cases} 1 & \text{if } e = (v, w) \text{ and } v < w \\ -1 & \text{if } e = (v, w) \text{ and } v > w \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Example 2 *The Laplacian and the Incidence matrix of the graph G=*



is $L_G = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ $\quad \nabla_G = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$

Lemma 10 $L_G = \nabla^T \nabla$.

Proof Observe that $\left[\nabla^T \nabla\right]_{ij} = (i\text{th column of } \nabla) \cdot (j\text{th column of } \nabla) = \sum_e \left([\nabla]_{e,v_i}\right)\left([\nabla]_{e,v_j}\right)$ This gives three cases:

- When $i = j$,
$$\left[\nabla^T \nabla\right]_{ij} = \sum_e \left([\nabla]_{e,v_i}\right)^2 = \sum_{e \text{ incident to } v_i} 1 = deg(i).$$
- When $i \neq j$ and no edge exists between v_i and v_j,
$$\left[\nabla^T \nabla\right]_{ij} = \sum_e \left([\nabla]_{e,v_i}\right)\left([\nabla]_{e,v_j}\right) = 0$$
as every edge is non-incident to at least one of v_i, v_j.
- When $i \neq j$ and exists an edge e' between v_i and v_j,
$$\left[\nabla^T \nabla\right]_{ij} = \sum_e \left([\nabla]_{e,v_i}\right)\left([\nabla]_{e,v_j}\right) = \left([\nabla]_{e',v_i}\right)\left([\nabla]_{e',v_j}\right) = -1. \quad \blacksquare$$

Corollary 11 *Note that*
$$x^T L_G x = ||\nabla x||^2 = \sum_{(i,j)\in E} (x_i - x_j)^2,$$
This gives another proof that L is PSD.

3.5 Dimension of the Null Space

Theorem 12 *If G is connected, the null space is 1-dimensional and spanned by the vector* **1**.

Proof Let $x \in \text{null}(L)$, i.e. $L_G x = 0$. This implies
$$x^T L_G x = \sum_{(i,j)\in E} (x_i - x_j)^2 = 0.$$
Thus, $x_i = x_j$ for every $(i, j) \in E$. As G is connected, this means that all x_i are equal. Thus every member of the null space is a multiple of **1**. $\blacksquare$

Corollary 13 *If G is connected,* $\lambda_2 > 0$.

Corollary 14 *The dimension of the null space of* L_G *is exactly the number of connected components of* G.

4 Spectra of Some Common Graphs

We compute the spectra of some graphs:

Lemma 15 (Complete graph) *The Laplacian for the complete graph K_n on n vertices has eigenvalue 0 with multiplicity 1 and eigenvalue n with multiplicity $n-1$ and associated eigenspace $\{x|x \cdot \mathbf{1} = 0\}$.*

Proof By corollary 13, we conclude that eigenvalue 0 has multiplicity 1. Now, take any vector v which is orthogonal to $\mathbf{1}$ and consider $[L_{K_n} v]_i$. Note that this value is equal to

$$(n-1)v_i - \sum_{j \neq i} v_j = nv_i - \sum_j v_j = nv_i$$

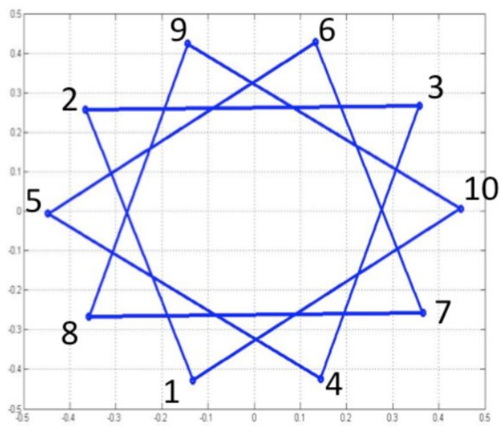
Hence any vector v which is orthogonal to $\mathbf{1}$ is an eigenvector with an eigenvalue n. ■

Lemma 16 (Ring graph) *The Laplacian for the ring graph R_n on n vertices has eigenvectors*

$$\begin{aligned} x_k(u) &= \sin(2\pi k u/n), \text{ and} \\ y_k(u) &= \cos(2\pi k u/n) \end{aligned}$$

for $0 \leq k \leq n/2$. Both x_k and y_k have eigenvalue $2 - 2\cos(2\pi k/n)$. Note that, $x_0 = \mathbf{0}$ should be ignored and y_0 is $\mathbf{1}$, and when n is even $x_{n/2} = \mathbf{0}$ should be ignored and we only have $y_{n/2}$.

Proof . The best way to see is to plot the graph on the circle using these vectors as coordinates. Below is the plot for a $k = 3$.



Just consider vertex 1. Keep in mind that $\sin(2x) = 2\sin(x)\cos(x)$. Then,

$$\begin{aligned} [Lx_k]_1 &= 2x_k(1) - x_k(0) - x_k(2) \\ &= 2\sin(2\pi k/n) - 0 - \sin(2\pi k 2/n) \\ &= 2\sin(2\pi k/n) - 2\sin(2\pi k/n)\cos(2\pi k/n) \\ &= (2 - 2\cos(2\pi k/n))\sin(2\pi k/n) \\ &= (2 - 2\cos(2\pi k/n))x_k(1) \end{aligned}$$

■

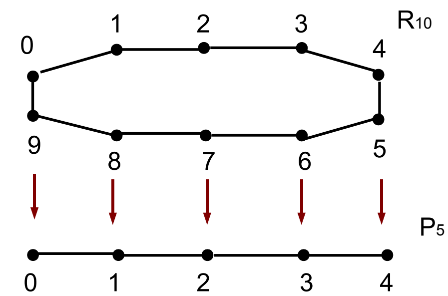
Note that this shows that $x(u) = \Re(e^{2\pi i(ku+c)/n})$ is an eigenvector for any $k \in \mathbb{Z}, c \in \mathbb{C}$.

Lemma 17 (Path graph) *The Laplacian for the path graph P_n on n vertices has the same eigenvalues as R_{2n} and eigenvectors*

$$v_k(u) = \sin(\pi k u/n + \pi/2n)$$

for $0 \leq k < n$

Proof



We will realize P_n as quotient of R_{2n}. Suppose z was an eigenvector of $L_{R_{2n}}$ in which $z_i = z_{2n-1-i}$ for $0 \leq i < n$. Take the first n components of z and call this vector v. Note that for $0 < i < n$:

$$\begin{aligned}
[L_{P_n}v]_i &= 2(v_i - \sum \text{neighbors of i in } P_n) \\
&= 2(z_i - \sum \text{neighbors of i in } R_{2n}) \\
&= (z_i - \sum \text{neighbors of i in } R_{2n}) + (z_{2n-i-1} - \sum \text{neighbors of } (2n-i-1)\ in\ R_{2n}) \\
&= \frac{1}{2}([L_{R_{2n}}z]_i + [L_{R_{2n}}z]_{2n-i-1}) \\
&= \frac{1}{2}(\lambda z_i + \lambda z_{2n-i-1}) \\
&= \lambda z_i \\
&= \lambda v_i
\end{aligned}$$

Now consider the case when $i = 0$.

$$\begin{aligned}
[L_{P_n}v]_0 &= v_0 - v_1 \\
&= 2v_0 - v_1 + v_0 \\
&= 2z_0 - z_1 + z_0 \\
&= 2z_0 - z_1 + z_{2n-1} \\
&= \lambda z_0 \\
&= \lambda v_0
\end{aligned}$$

Hence, v is an eigenvector of L_{P_n}. Now we show that such v exists, that is, there exists eigenvector z of $L_{R_{2n}}$ in which $z_i = z_{2n-1-i}$ for $0 \leq i < n$. Take z,

$$\begin{aligned}
z_k(u) &= \sin(\pi k u/n + \pi/2n) \\
&= \sin(\pi k u/n)\cos(\pi/2n) + \cos(\pi k u/n)\sin(\pi/2n) \\
&= x_k(u)\cos(\pi/2n) + y_k \sin(\pi/2n)
\end{aligned}$$

We see that z_k is in the span of x_k and y_k. Hence, z_k is an eigenvector of $L_{R_{2n}}$ with an eigenvalue $2 - 2\cos(2\pi k/n)$ by lemma 16. Check that z_k satisfies $z_k(i) = z_k(2n-1-i)$ ■

4.1 Graph Products

The next natural example is the grid graph, which will follow from general theory about product graphs.

Definition 18 *Let $G = (V, E)$, and $H = (W, F)$. The* product graph *$G \times H$ has vertex set $V \times W$ and edge set:*

$$((v_1, w), (v_2, w)), \quad \forall (v_1, v_2) \in E, w \in W \qquad \text{and}$$
$$((v, w_1), (v, w_2)), \quad \forall (w_1, w_2) \in F, v \in V.$$

Example 3 *$P_n \times P_m = G_{n,m}$. We see that the vertices of $P_n \times P_m$ are:*

$$v(G_{n,m}) = \left\{ \begin{array}{cccc} (v_1, w_1) & (v_1, w_2) & \cdots & (v_1, w_m) \\ (v_2, w_1) & (v_2, w_2) & \cdots & (v_2, w_m) \\ \vdots & \vdots & \ddots & \vdots \\ (v_n, w_1) & (v_n, w_2) & \cdots & (v_n, w_m) \end{array} \right\}$$

The vertices are written in the above layout because it makes the edges intuitive. The edges are:

- *For a fixed w, i.e. a column in the above layout, a copy of P_n.*
- *For a fixed v, i.e. a row in the above layout, a copy of P_m.*

Thus $P_n \times P_m = G_{n,m}$, which is to say the product of two path graphs is the grid graph.

Theorem 19 (Graph Products) *If L_G has eigenvectors $v_1, \ldots, v_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, and L_H has eigenvectors $w_1, \ldots, w_k$ with eigenvalues $\mu_1, \ldots, \mu_k$, then $L_{G \times H}$ has, for all $1 \le i \le n$, $1 \le j \le k$, an eigenvector:*

$$z_{ij}(v, w) = x_i(v) y_j(w)$$

of eigenvalue $\lambda_i + \mu_j$.

Note importantly that eigenvalues add here, they do not multiply.
Proof Let A_m be the graph with m isolated vertices. We can then decompose the product as:

$$G \times H = (G \times A_k) \cup (A_n \times H),$$

i.e. the edge union of k disjoint copies of G and n disjoint copies of H, exactly as in the definition. By Lemmas 1 and 3 we have

$$L_{G \times H} = L_{G \times A_k} + L_{A_n \times H} = L_G \otimes I_k + I_n \otimes L_H$$

Consider $z_{ij} = x_i \otimes y_j$ as above for a fixed i and j, we see that:

$$\begin{aligned} L_{G \times H} z_{ij} &= (L_G \otimes I_k)(x_i \otimes y_j) + (I_n \otimes L_H)(x_i \otimes y_j) \\ &= (\lambda_i x_i \otimes y_j) + (x_i \otimes \mu_j y_j) \\ &= (\lambda_i + \mu_j)(x_i \otimes y_j) = (\lambda_i + \mu_j) z_{ij}. \quad \blacksquare \end{aligned}$$

Corollary 20 *$G_{n,m}$ has eigenvectors and eigenvalues completely determined by those of P_n and P_m as above.*

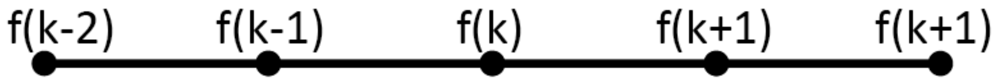
5 Why is this called the Laplacian?

It turns out that the graph Laplacian is very naturally related to the continuous Laplacian.

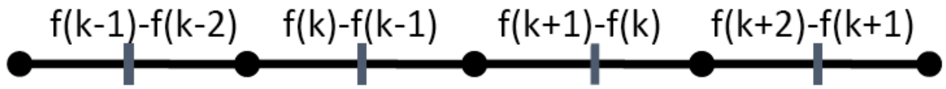
- In 1 dimension, the continuous Laplacian is $\frac{d}{dx}$.
- In 2 dimensions, the continuous Laplacian is $\nabla f = \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2}$.

5.1 Discretizing Derivatives, 1d case

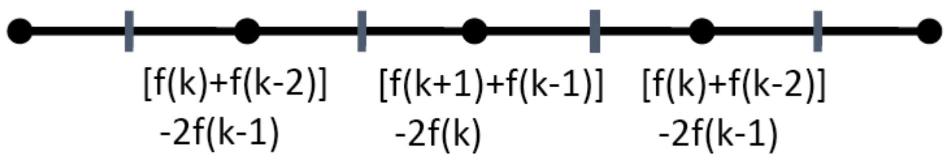
Consider a 1d function $f : \mathbb{R} \to \mathbb{R}$, which we wish to discretize at the points $(\ldots, k-2, k-1, k, k+1, k+2, \ldots)$:



We approximate the first derivative at the line midpoints, $\frac{df}{dx}$, up to scaling, by taking the differences between the values at the adjacent points:



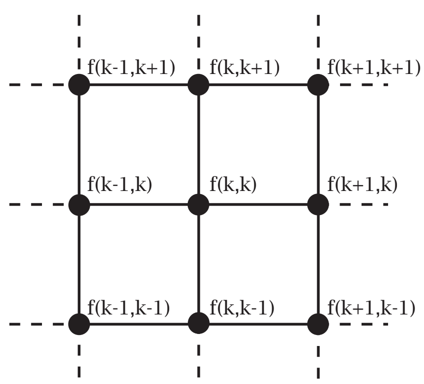
The discrete first derivative of f is a function on edges, and is, up to scaling $\nabla_{P_n} f$, the incidence matrix of P_n defined earlier. In order to compute the second derivative at the original points, $\frac{d^2 f}{dx^2}$, again up to scaling, we take the differences of the adjacent midpoints at the vertices:



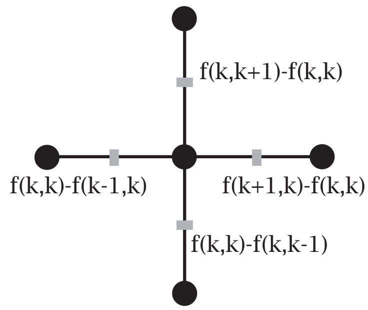
The discrete second derivative of f is thus, up to scaling, $-L_{P_n} f$.

5.2 Discretizing Derivatives, 2d case

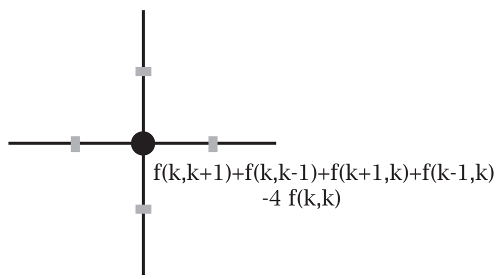
Here we discretize $f : \mathbb{R}^2 \to \mathbb{R}$ on a grid:



To compute the discrete derivative in the x and y directions, we'll just look at a little piece. On the horizontal edges, we approximate $\frac{df}{dx}$ up to scaling, and do likewise on the vertical edges with $\frac{df}{dy}$:



Again, the discrete derivative of f is a function on edges. When we consider the concatenation of the two discretization of the directional derivatives, we see that the discretization of the gradient, up to scaling, is $\nabla_{G_{n,m}} f$. Finally we use this to compute the discretized Laplacian, up to scaling, and get:



Thus the discretized Laplacian in two dimensions of f is $-L_{G_{n,m}} f$.

5.3 A Note on Fourier Coefficients

We observed that paths and rings had eigenvectors that looked like Fourier coefficients. In the continuous case:

$$\begin{aligned} \frac{d^2 \sin(kx+c)}{dx^2} &= k^2 \sin(kx+c) \\ \frac{d^2 \cos(kx+c)}{dx^2} &= k^2 \cos(kx+c) \end{aligned}$$

Thus $\sin(kx+c)$ and $\cos(kx+c)$ are eigenfunctions of the $\frac{d^2}{dx^2}$ operator, i.e. the 1d Laplacian, both with eigenvalue k^2.

6 Bounding Laplacian Eigenvalues

Lemma 21 (Sum of the eigenvalues) *Given an n-vertex graph G with degrees d_i, where $d_{\max} = \max_i d_i$, and Laplacian L_G with eigenvalues λ_i,*

$$\sum_i \lambda_i = \sum_i d_i \leq d_{\max} n \tag{8}$$

Proof The first two expressions are both the trace, the upper bound is trivial. ■

Lemma 22 (Bounds on λ_2 and λ_n) *Given λ_i and d_i as above,*

$$\lambda_2 \leq \frac{\sum_i d_i}{n-1} \tag{9}$$

$$\lambda_n \geq \frac{\sum_i d_i}{n-1} \tag{10}$$

Proof By the previous slide and the fact that $\lambda_1 = 0$, we get $\sum_{i=2}^n \lambda_i = \sum_i d_i$. As $\lambda_2 \leq \cdots \leq \lambda_n$, the bounds follow immediately. ■

7 Bounding λ_2 and $\lambda_{\max}$

Theorem 23 (Courant-Fischer Formula) *For any $n \times n$ symmetric matrix A,*

$$\begin{aligned}
\lambda_1 &= \min_{||x||=1} x^T A x = \min_{x \neq 0} \frac{x^T A x}{x^T x} \\
\lambda_2 &= \min_{\substack{||x||=1 \\ x \perp v_1}} x^T A x = \min_{\substack{x \neq 0 \\ x \perp v_1}} \frac{x^T A x}{x^T x} \\
\lambda_{\max} &= \max_{||x||=1} x^T A x = \max_{x \neq 0} \frac{x^T A x}{x^T x}
\end{aligned} \tag{11}$$

Proof We consider the diagonalization $A = Q^T \Lambda Q$. As seen earlier, $x^T A x = (Qx)^T \Lambda (Qx)$. As Q is orthogonal, we also have $||Qx|| = x$. Thus it suffices to consider diagonal matrices. Moreover, all of the equalities on the right follow immediately from linearity. Thus we need to consider, for $||x|| = 1$:

$$x^T \Lambda x = (x_1 \cdots x_n) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum \lambda_i x_i^2 = \frac{\sum \lambda_i x_i^2}{\sum x_i^2}.$$

We compute the gradient and find:

$$\left[\nabla_x \left(x^T \Lambda x\right)\right]_i = \frac{2\lambda_i x_i}{\sum x_i^2} - \frac{2\lambda_i x_i^3}{\left(\sum x_i^2\right)^2} = 2\lambda_i (x_i - x_i^3)$$

thus all extremal values occur when one $x_i = 1$ and the rest are 0. The identities follow immediately. ■

Corollary 24 (Rayleigh Quotient) *Letting $G = (V, E)$ be a graph with Laplacian L_G,*

$$\begin{aligned}
\lambda_1 &= 0 \qquad v_1 = \mathbf{1} \\
\lambda_2 &= \min_{\substack{x \perp v_1 \\ x \neq 0}} \frac{x^T L_G x}{x^T x} = \min_{\substack{\sum x = 0 \\ x \neq 0}} \frac{\sum_{(i,j) \in E} (x_i - x_j)^2}{\sum_{i \in V} x_i^2} \\
\lambda_{\max} &= \max_{x \neq 0} \frac{x^T L_G x}{x^T x} = \max_{x \neq 0} \frac{\sum_{(i,j) \in E} (x_i - x_j)^2}{\sum_{i \in V} x_i^2}
\end{aligned} \tag{12}$$

The Rayleigh Quotient is a useful tool for bounding graph spectra. Whereas before we had to consider all possible vectors x, now in order to get an upper bound on λ_2 we need only produce a vector with small Rayleigh quotient. Likewise to get a lower bound on $\lambda_{\max}$ we need only to find a vector with large Rayleigh quotient.

Examples next lecture!

18.409 Topics in Theoretical Computer Science: An Algorithmist's Toolkit
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