18.409 An Algorithmist's Toolkit September 10, 2009

Lecture 1

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1 Overview

The class's goals, requirements, and policies were introduced, and topics in the class were described. Everything in the overview should be in the course syllabus, so please consult that for a complete description.

2 Linear Algebra Review

This course requires linear algebra, so here is a quick review of the facts we will use frequently.

Definition 1 *Let M by an $n \times n$ matrix. Suppose that*

$$Mx = \lambda x$$

for $x \in \mathbb{R}^n$, $x \neq 0$, and $\lambda \in \mathbb{R}$. Then we call x an eigenvector *and λ an* eigenvalue *of M.*

Proposition 2 *If M is a symmetric $n \times n$ matrix, then*

- *If v and w are eigenvectors of M with different eigenvalues, then v and w are orthogonal ($v \cdot w = 0$).*
- *If v and w are eigenvectors of M with the same eigenvalue, then so is $q = av + bw$, so eigenvectors with the same eigenvalue need not be orthogonal.*
- *M has a full orthonormal basis of eigenvectors $v_1, \ldots, v_n$. All eigenvalues and eigenvectors are real.*
- *M is diagonalizable:*
$$M = V \Lambda V^T$$
where V is orthogonal ($VV^T = I_n$), with columns equal to $v_1, \ldots, v_n$, and Λ is diagonal, with the corresponding eigenvalues of M as its diagonal entries. So $M = \sum_{i=1}^{n} \lambda_i v_i v_i^T$.

In Proposition 2, it was important that M was symmetric. No results stated there are necessarily true in the case that M is not symmetric.

Definition 3 *We call the span of the eigenvectors with the same eigenvalue an* eigenspace*.*

3 Matrices for Graphs

During this course we will study the following matrices that are naturally associated with a graph:

- The Adjacency Matrix
- The Random Walk Matrix
- The Laplacian Matrix
- The Normalized Laplacian Matrix

Let $G = (V, E)$ be a graph, where $|V| = n$ and $|E| = m$. We will for this lecture assume that G is unweighted, undirected, and has no multiple edges or self loops.

Definition 4 *For a graph G, the* adjacency matrix $A = A_G$ *is the $n \times n$ matrix given by*

$$A_{i,j} = \begin{cases} 1 & \textit{if } (i,j) \in E \\ 0 & \textit{otherwise} \end{cases}$$

For an unweighted graph G, A_G is clearly symmetric.

Definition 5 *Given an unweighted graph G, the* Laplacian matrix $L = L_G$ *is the $n \times n$ matrix given by*

$$L_{i,j} = \begin{cases} -1 & \textit{if } (i,j) \in E \\ d_i & \textit{if } i = j \\ 0 & \textit{otherwise} \end{cases}$$

where d_i is the degree of the i^{th} vertex.

For unweighted G, the Laplacian matrix is clearly symmetric. An equivalent definition for the Laplacian matrix is

$$L_G = D_G - A_G,$$

where D_G is the diagonal matrix with i^{th} diagonal entry equal to the degree of v_i, and A_G is the adjacency matrix.

4 Example Laplacians

Consider the graph H with adjacency matrix

$$\mathbf{A_H} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

This graph has Laplacian

$$\mathbf{L_H} = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

Now consider the graph G with adjacency matrix

$$\mathbf{A_G} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

This graph has Laplacian

$$\mathbf{L_G} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

L_G is a matrix, and thus a linear transformation. We would like to understand how L_G acts on a vector v. To do this, it will help to think of a vector $v \in \mathbb{R}^3$ as a map $X : V \to \mathbb{R}$. We can thus write v as

$$\mathbf{v} = \begin{pmatrix} X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

The action of L_G on v is then

$$\mathbf{L_G} v = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} X(1) \\ X(2) \\ X(3) \end{pmatrix} = \begin{pmatrix} X(1) - X(2) \\ 2X(2) - X(1) - X(3) \\ X(3) - X(2) \end{pmatrix} = \begin{pmatrix} X(1) - X(2) \\ 2\left(X(2) - [\frac{X(1)+X(3)}{2}]\right) \\ X(3) - X(2) \end{pmatrix}$$

For a general Laplacian, we will have

$$[L_G v]_i = [d_i * (X(i) - \text{ average of X on neighbors of i})]$$

Remark For any G, $\mathbf{1} = (1, \ldots, 1)$ is an eigenvector of L_G with eigenvalue 0, since for this vector $X(i)$ always equals the average of its neighbors' values.

Proposition 6 *We will see later the following results about the eigenvalues λ_i and corresponding eigenvectors v_i of L_G:*

- *Order the eigenvalues so $\lambda_1 \leq \ldots \leq \lambda_n$, with corresponding eigenvectors $v_1, \ldots, v_n$. Then $v_1 = \mathbf{1}$ and $\lambda_1 = 0$. So for all i $\lambda_i \geq 0$.*
- *One can get much information about the graph G from just the first few nontrivial eigenvectors.*

5 Matlab Demonstration

As remarked before, vectors $v \in \mathbb{R}^n$ may be construed as maps $X_v : V \to \mathbb{R}$. Thus each eigenvector assigns a real number to each vertex in G. A point in the plane is a pair of real numbers, so we can embed a connected graph into the plane using $(X_{v_2}, X_{v_3}) : V \to \mathbb{R}^2$. The following examples generated in Matlab show that this embedding provides representations of some planar graphs.

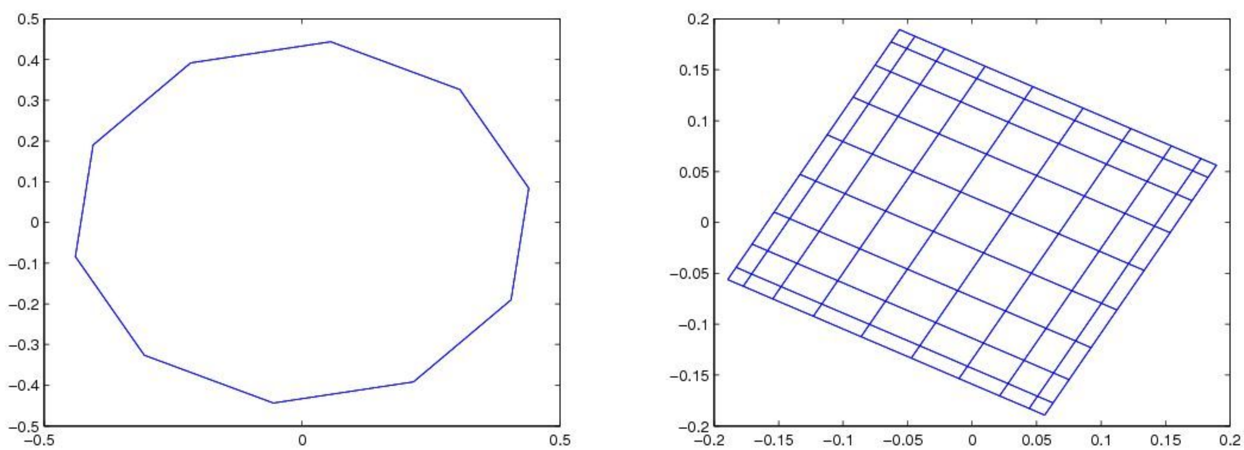

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Figure 1: Plots of the first two nontrivial eigenvectors for a ring graph and a grid graph

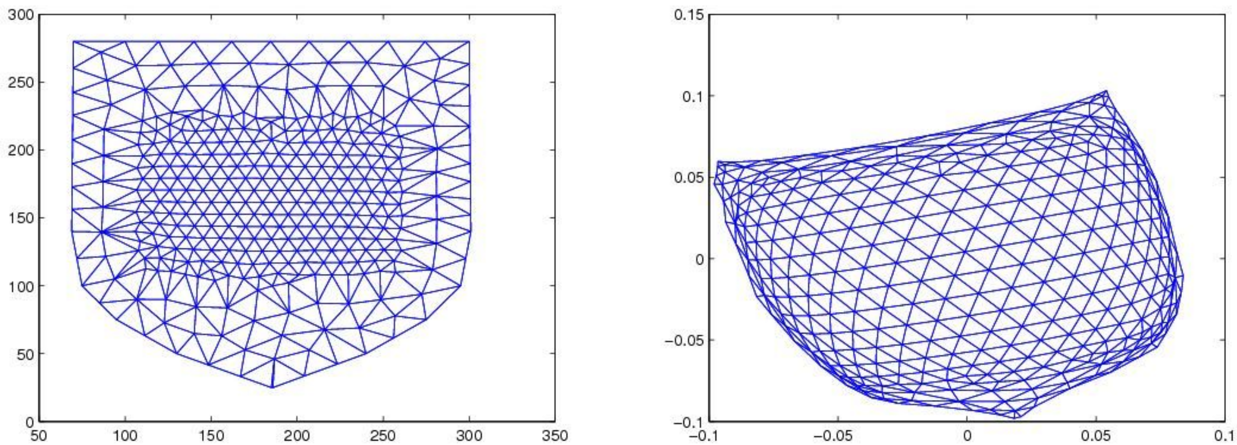

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Figure 2: Handmade graph embedding (left) and plot of the first two nontrivial eigenvectors (right) for an interesting graph due to Spielman

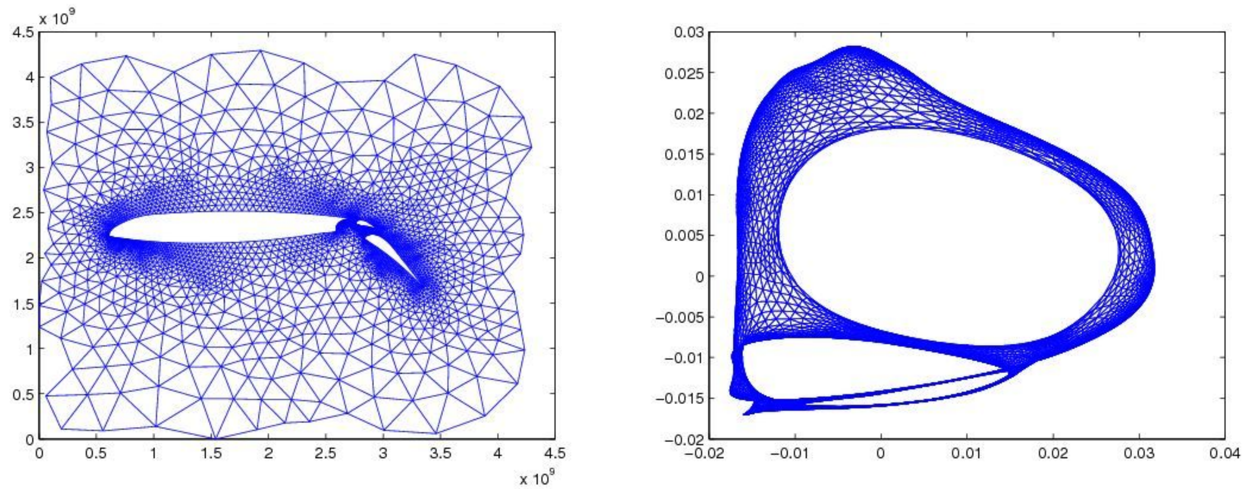

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Figure 3: Handmade graph embedding (left) and plot of first two nontrivial eigenvectors (right) for a graph used to model an airfoil

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