18.409 The Behavior of Algorithms in Practice 2/26/2002

Lecture 5

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Smoothed Complexity of Gaussian Elimination

Today we will show that the smoothed complexity of solving an $n \times n$ linear system to t bits of accuracy, using Gaussian Elimination without pivoting, is $O(n^3(\log(n/\sigma) + t))$.

More formally, we want to prove the following result.

Theorem 1 *Let $\bar{A}$ be any matrix with $\|\bar{A}\|_\infty \leq 1$ and let $A = \bar{A} + G$ where G is a Gaussian random matrix with variance $\sigma^2 \leq 1$. Then the expected number of digits needed to solve $Ax = b$ to t bits of accuracy is $O(log(n/\sigma) + t)$.*

We will prove this theorem via a sequence of lemmas.

Recall that if we run Gaussian Elimination with ϵ_{mach} precision then we get $\hat{x}$ s.t. $(A + \delta A)\hat{x} = b$ and

$$\frac{\|\delta A\|_\infty}{\|A\|_\infty} \leq n\epsilon_{mach}\left(3 + \frac{5\|L\|_\infty\|U\|_\infty}{\|A\|_\infty}\right)$$

Also,

$$\frac{\|\hat{x} - x\|}{\|x\|} \leq \kappa(A)\frac{\|\delta A\|_\infty}{\|A\|_\infty}$$

In the last class, we showed that

$$\frac{\|U\|_\infty}{\|A\|_\infty} \leq \max_k \frac{\|A^{(k)}\|_\infty}{\|A\|_\infty}$$

where $A^{(k)}$ is the $(n-k-1)$x$(n-k-1)$ matrix after first k eliminations.
We also proved that

$$\frac{\|A^{(k)}\|_\infty}{\|A\|_\infty} \leq n\|A^{-1}_{1:k,1:k}\| \max(\|A_{1:k,(k+1):n}\|_\infty, \|A_{(k+1):n,1:k}\|_\infty)$$

and

$$Pr[\|A^{-1}_{1:k,1:k}\| > x] \leq \frac{k^{3/2}}{x} \tag{1}$$

Fact: For a Gaussian random variable, G, with mean 0 and variance 1, $Pr[G > x] \leq \frac{e^{-x^2/2}}{\sqrt{2\pi}x}$

Using this fact together with the union bound, we get:

$$Pr[\max(\|A_{1:k,(k+1):n}\|_\infty, \|A_{(k+1):n,1:k}\|_\infty) > x] \leq \frac{n^2}{2}\frac{e^{-x^2/2}}{\sqrt{2\pi}x} \tag{2}$$

In order to combine the probability bounds from (1) and (2) we use the following combination lemma.

Lemma 2 (Combination Lemma 1) *Let A and B be independent random variables s.t.*

$$Pr[A > x] \leq \frac{\alpha}{x}$$

$$Pr[B > x] \leq \frac{\beta e^{-x^2/2}}{x}$$

Then,

$$Pr[AB > x] \leq \frac{\alpha(\sqrt{2\log(\beta)})}{x}$$

Proof $AB > x \Rightarrow A > x/\sqrt{2\log\beta}$ or $\exists i \geq \sqrt{2\log\beta}$ s.t. $i \leq B \leq i+1$ and $A \geq x/i+1$

Therefore,

$$\begin{aligned} Pr[AB > x] &\leq Pr[A \geq x/\sqrt{2\log(\beta)}] + \sum_{i \geq \sqrt{2\log(\beta)}} Pr[A \geq x/i+1] Pr[B \geq i] \\ &\leq \frac{\alpha\sqrt{2\log(\beta)}}{x} \end{aligned}$$

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Since the random variables in $A^{-1}_{1:k,1:k}$ and $A_{1:k,k+1:n}, A_{k+1:n,1:k}$ are independent we can use the combination lemma to get the following bound:

$$Pr[\frac{\|A^{(k)}\|_\infty}{\|A\|_\infty} \geq nx] \leq \frac{n^{3/2}}{x}(2\sqrt{\log n} + 4) \tag{3}$$

Finally, the union bound lets us get the following bound on the tail probability of the growth factor, $\frac{\|U\|_\infty}{\|A\|_\infty}$

$$Pr[\frac{\|U\|_\infty}{\|A\|_\infty} \geq x] \leq \frac{n^{7/2}}{x}(2\sqrt{\log n} + 4) \tag{4}$$

Next we need to bound $\|L\|_\infty$.

Exercise: For $j > k$, $L_{j,k} = \frac{a_{j,k}^{k-1}}{a_{k,k}^{k-1}}$

Now,

$$a_{k,k}^{(k-1)} = \bar{a}_{k,k} + g_{k,k} - a_{k,1:k-1} A_{1:k-1,1:k-1}^{-1} a_{1:k-1,k}$$

And since the $g_{k,k}$ are chosen independently,

$$Pr[|a_{k,k}^{(k-1)}| < \epsilon] \leq \epsilon/\sigma$$

Equivalently, since the $g_{k,k}$ do not appear in $a_{j,k}^{(k-1)}, j > k$

$$Pr[|a_{k,k}^{(k-1)}| < \epsilon \,|\{a_{j,k}^{k-1}\}, j > k] \leq \epsilon/\sigma$$

which can be rewritten as:

$$Pr[\frac{1}{|a_{k,k}^{(k-1)}|} > x \mid a_{j,k}^{k-1}, j > k] \leq \frac{1}{/}\sigma x \tag{5}$$

Exercise: Show that

$$Pr[\exists j : |a_{j,k}^{(k-1)} > x] \leq \frac{n^{5/2}(2\sqrt{\log n} + 4)}{x} \tag{6}$$

Lemma 3 (Combination Lemma 2) *Let A and B be random variables satisfying:*

$$Pr[A > x] \leq \alpha/x$$

$$Pr[B > x|A] \leq \beta/x$$

Then

$$Pr[AB > x] \leq \frac{2\alpha\beta\lceil\log x\rceil + \alpha + \beta}{x}$$

Proof

$AB > x \Rightarrow$ either $A > x$ or $B > x$ or $\exists i, 1 \leq i \leq \lceil\log x\rceil s.t. A \geq 2^{i-1}\, and\, B \geq x/2^i$

It follows that:

$$\begin{aligned}
Pr[AB > x] &\leq Pr[A > x] + Pr[B > x] + \sum_{i=1}^{\lceil\log x\rceil} Pr[A \geq 2^{i-1}\, and\, B \geq x/2^i] \\
&\leq \alpha/x + \beta/x + \sum_i Pr[A \geq 2^{i-1}] Pr[B \geq x/2^i | A \geq 2^{i-1}] \\
&\leq \frac{\alpha+\beta}{x} + \sum_i \frac{\alpha}{2^i}\frac{\beta 2^i}{x} \\
&\leq \frac{\alpha + \beta + 2\alpha\beta\lceil\log x\rceil}{x}
\end{aligned}$$

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The above combination lemma let's us combine the bounds in (5) and (6) to get the following bound on $\|L\|_\infty$:

$$Pr[\|L\|_\infty \geq 5n^{7/2}(\sqrt{\log n}+1)x\log x/\sigma] \leq 1/x$$

So far we've proved the following:

$$Pr[\|L\|_\infty \geq 5n^{7/2}(\sqrt{\log n}+1)x\log x/\sigma] \leq 1/x$$

$$Pr[\frac{\|U\|_\infty}{\|A\|_\infty} \geq 2n^{7/2}(\sqrt{\log n}+1)x/\sigma] \leq 1/x$$

$$Pr[\|A\| \geq n^{1/2}(1+\sqrt{4\log x/n}] \leq 1/x$$

$$Pr[\|A^{-1}\| \geq n^{3/2}x/\sigma] \leq 1/x$$

Combining everything, we get:

$$Pr[\frac{\kappa(A)\|L\|_\infty\|U\|_\infty}{\|A\|_\infty} \geq 10n^9(\sqrt{\log n}+1)^2(1+\sqrt{4\log x/n})x^3/\sigma^3] \leq 4/x$$

In order to get an estimate of the digits we need a statment about the log of $\frac{\kappa(A)\|L\|_\infty\|U\|_\infty}{\|A\|_\infty}$.

Exercise: If $Pr[a > \alpha x^k] \leq 1/x$ then $E[\log(a)] \leq k\log(\alpha) + f(k)$ where $f(k) \leq (\frac{1}{1-2^{-1/k}})^2$

Using this fact let's us claim the desired result, viz.,

$$E[\log(\frac{\kappa(A)\|L\|_\infty\|U\|_\infty}{\|A\|_\infty})] \leq O(\log(n/\sigma))$$

Drawbacks of this analysis

This analysis is limited to the case when no pivoting is done. It would be desirable to prove something about partial pivoting. It seems that we should be able to get a high probability result with exponentially instead of polynomially small probability for this case. Experiments seem to validate this hypothesis too.