18.409 The Behavior of Algorithms in Practice	March 19, 2002
Lecture 11	
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Linear Programming

We now start with the study of smoothed analysis for linear programming. In the forthcoming lectures we shall cover three methods for LP:

1.- Elementary/Naive/Stupid methods (this lecture).

2.- Simplex method (part in this lecture).

3.- Interior point method.

Introduction

Given $a_1, \ldots, a_n \in \mathbb{R}^d$ and $c \in \mathbb{R}^d$, Dan's favorite linear program is the following

$$\begin{aligned} \text{[LP1]} \quad \max \quad & \alpha \\ \text{s.t.} \quad & \alpha c \in \mathrm{ch}(a_1, \ldots, a_n), \\ & \alpha \geq 0. \end{aligned}$$

where $\mathrm{ch}(a_1, \ldots, a_n)$ denotes the convex hull defined by the vectors $a_1, \ldots, a_n$. We know that $c \in \mathrm{ch}(a_1, \ldots, a_n)$ if and only if there exists $y_1, \ldots, y_n \geq 0$ such that $\sum_{i=1}^n y_i = 1$ and $c = \sum_{i=1}^n y_i a_i$. Hence we can write [LP1] in an equivalent form

$$\begin{aligned} \text{[LP2]} \quad \max \quad & \alpha \\ \text{s.t.} \quad & \exists y_1 \ldots y_n \geq 0 \\ & \sum_{i=1}^n y_i = 1, \sum_{i=1}^n y_i a_i = \alpha c. \end{aligned}$$

By letting $y'_i = \frac{y_i}{\alpha}$, we have that $y'_i \geq 0$, $\sum_{i=1}^n y'_i = \frac{1}{\alpha}$ and $c = \sum_{i=1}^n y'_i a_i$. Then, maximizing α is the same as minimizing $\sum_{i=1}^n y'_i$, and hence [LP2] becomes

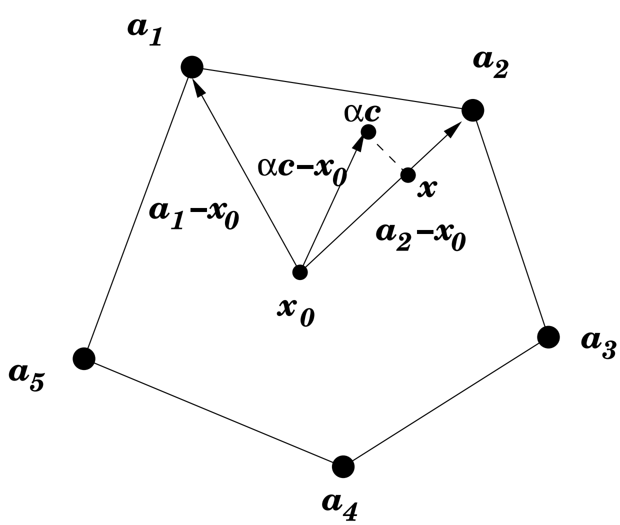
$$\begin{array}{lll} \text{[LP3]} & \min & \sum_{i=1}^n y'_i \\ & \text{s.t.} & \sum_{i=1}^n y'_i a_i = c, \\ & & y'_i \geq 0. \end{array}$$

1.- Von Neumann's algorithm for LP

We know see an elementary algorithm to solve [LP1]. The algorithm will use binary search on α. And it will call a subroutine which decides whether a given αc belongs to $\mathrm{ch}(a_1, \ldots, a_n)$, this is called the *decision problem*.

To solve the decision problem Von Neumann's algorithm proceeds as follows:

- Take $x_0 \in \mathrm{ch}(a_1, \ldots, a_n)$, say $x_0 = \frac{1}{n}\sum_{i=1}^n a_i$.
- Choose i maximizing $\langle(\alpha c - x_0), (a_i - x_0)\rangle$ ($i = 2$ in the figure below).
- Find the point x from x_0 to a_i (i.e. $x \in \mathrm{ch}(x_0, a_i)$) closest to αc.
- $x_0 = x$ and repeat.



This procedure converges since whenever $\alpha c \in \mathrm{ch}(a_1, \ldots, a_n)$ we have $\max_i\{\langle(\alpha c - x_0), (a_i - x_0)\rangle\} \geq 0$ for some i. Otherwise, $\langle(\alpha c - x_0), (a_i - x_0)\rangle < 0$ for all a_i. This implies that there is an hyperplane separating x_0 form $\{a_1, \ldots, a_n\}$, from where $\alpha c \notin \mathrm{ch}(a_1, \ldots, a_n)$.

Theorem 1 (Dantzig). *Von Neumann's Algorithm obtains a point x_0 such that $||x_0 - \alpha c|| < \epsilon$ in*

$$\frac{4\max\{||\alpha c||, \max_i ||a_i||\}}{\epsilon^2}$$

iterations.

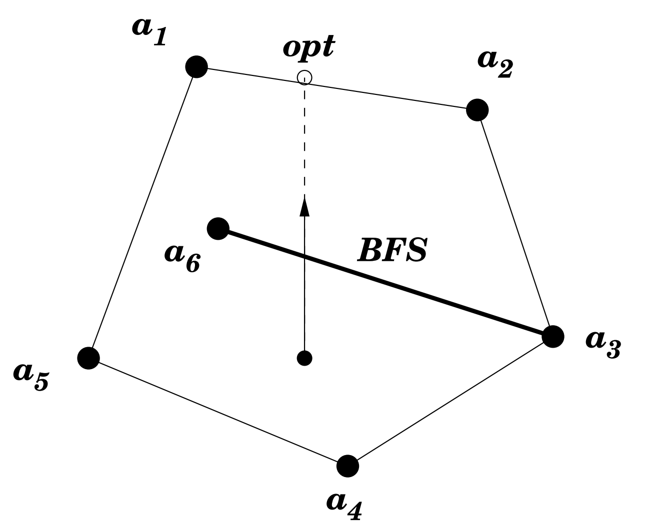
Theorem 2 (Freund-Epelman). *Von Neumann's Algorithm obtains a point x_0 such that $||x_0 - \alpha c|| < \epsilon$ in*

$$\frac{8\max\{||\alpha c||, \max_i ||a_i||\}}{r^2}\log\left(\frac{||x_0 - \alpha c||}{\epsilon}\right)$$

iterations. Where $r = \mathrm{distance}(||\alpha c||, \mathrm{boundary}(\mathrm{ch}(a_1, \ldots a_n)))$. r is a condition number of the linear program.

2.- The Simplex method

Consider again the linear program [LP1]. A basic feasible solution of [LP1] is collection of d points, $B \subset \{a_1, \ldots, a_n\}$ ($|B| = d$) such that $\alpha c \in \mathrm{ch}(B)$ for some $\alpha \geq 0$.



The simplex method proceeds as follows:

- Find a Basic feasible solution B.
- Find point, say a, in $\{a_1, \ldots, a_n\}$ above (with respect to c) the hyperplane $\mathrm{ch}(B)$.
- Remove one point, b, from $B \cup \{a\}$ so that $B' = B \cup \{a\} \setminus \{b\}$ is a basic feasible solution.
- $B = B \cup \{a\} \setminus \{b\}$ and repeat.

Initialization: Plant a Basic feasible solution. i.e. put d points very close to the origin (say at distance ϵ), so that they are not involved in an optimal solution. This is also known as the big M method since, $\min\{\sum_{i=1}^n y_i : \sum_{i=1}^n y_i a_i = c,\ y_i \geq 0\}$ can be written as

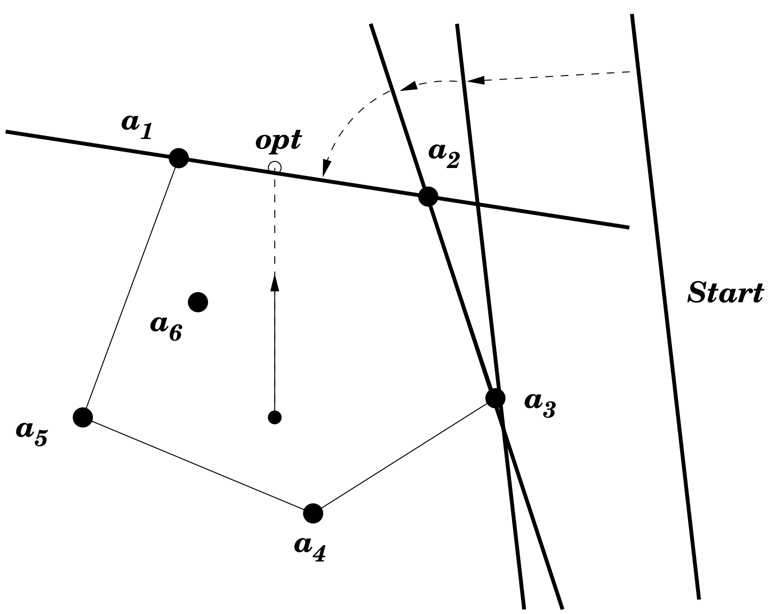
$$\begin{aligned} \text{[LP3]} \ \min \quad & \sum_{i=1}^n y_i + M \sum_{j=1}^d z_j \\ \text{s.t.} \quad & \sum_{i=1}^n y_i a_i + \sum z_j e_j = c, \\ & y_i \geq 0. \end{aligned}$$

Which again, by letting $M = 1/\epsilon$, is equivalent to

$$\begin{aligned} \text{[LP3]} \ \min \quad & \sum_{i=1}^n y_i + \sum_{j=1}^d z_j \\ \text{s.t.} \quad & \sum_{i=1}^n y_i a_i + \sum z_j (\epsilon e_j) = c, \\ & y_i \geq 0, \end{aligned}$$

and this is exactly the initialization method described above.

Duality: Another way of looking at Dan's linear program [LP1] is the following. Find a plane H and the minimum α such that, $\alpha c \in H$ and $\{a_1, \ldots, a_n\}$ are beneath H.



Since $H_x = \{a : \langle a, x \rangle = 1\}$ is the plane normal to a given vector x. The above problem, know as the *dual*, can be stated as

$$\begin{array}{lll} \text{[Dual-LP1]} & \min & \alpha \\ & \text{s.t.} & \langle \alpha c, x \rangle = 1 \\ & & \langle a_i, x \rangle \leq 1 \text{ for all } i. \end{array}$$

Now $\langle c, x \rangle = 1/\alpha$, hence the problem is simply

$$\begin{array}{lll} \text{[Dual-LP3]} & \max & \langle c, x \rangle \\ & \text{s.t.} & \langle a_i, x \rangle \leq 1 \text{ for all } i. \end{array}$$

We can conclude the following result.

Theorem 3. *If primal [LP1] has a solution, then the dual [Dual-LP1] has the same solution.*