

18.338J/16.394J: The Mathematics of Infinite Random Matrices Tridiagonal Matrices, Orthogonal Polynomials and the Classical Random Matrix Ensembles

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In class, we saw the connection between the so-called Hermite matrix and the semi-circular law. There is actually a deeper story that connects the classical random matrix ensembles to the classical orthogonal polynomials studied in classical texts such as [1] and more recent monographs such as [2]. We illuminate part of this story here. The website www.mathworld.com is an excellent reference for these polynomials and will prove handy when completing the exercises.

In any computational explorations, see if you can spot the interesting feature in the eigenvectors (either the first or last row/column) of the corresponding tridiagonal matrix.

1 Roots of orthogonal polynomials

Any weight function $w(x)$ on an interval $[a, b]$ (possibly infinite) defines a system of orthogonal polynomials π_n , $n = 0, 1, 2, \dots$, satisfying

$$\int_a^b \pi_m \pi_n w(x) dx = \delta_{mn}. \quad (1)$$

The polynomials can be generated easily, because they satisfy a three-term recurrence

$$\pi_0(x) = \left(\int_a^b w \right)^{-1/2}, \quad (2)$$

$$x\pi_n(x) = b_n\pi_{n-1}(x) + a_{n+1}\pi_n(x) + b_{n+1}\pi_{n+1}(x). \quad (3)$$

(Note that b_0 is taken to be zero.)

Perhaps surprisingly, the three-term recurrence can be used to find the roots of π_n as well. Simply form the symmetric tridiagonal matrix

$$T_n = \begin{bmatrix} a_1 & b_1 & & & & & \\ b_1 & a_2 & b_2 & & & & \\ & b_2 & a_3 & b_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & b_{n-2} & a_{n-1} & b_{n-1} & \\ & & & & b_{n-1} & a_n & \end{bmatrix}.$$

The roots of π_n are the eigenvalues of T_n !

Remark 1. *The symmetry of T follows from two choices which we made. First, in (1), π_n was normalized to have unit length under the inner product $\langle f, g \rangle = \int_a^b f g w$. Second, (3) was arranged in the form $x\pi_n = RHS$. Not all authors follow these conventions.*

To see that the eigenvalues are the roots, consider vectors of the form

$$v = (\pi_0(x), \pi_1(x), \dots, \pi_{n-1}(x))^T.$$

The recurrence (3) implies that $(Tv)_i = x\pi_{i-1}(x) = xv_i$ for $i = 1, \dots, n-1$, regardless of the value of x . Now, if x is a root of $\pi_n(x)$, then (3) specializes to

$$x\pi_{n-1}(x) = b_{n-1}\pi_{n-2}(x) + a_n\pi_{n-1}(x)$$

i.e.

$$xv_n = b_{n-1}v_{n-1} + a_nv_n = (Tv)_n.$$

Therefore, every root of $\pi_n(x)$ is an eigenvalue of T_n .

2 Hermite polynomials

Let $w(x) = e^{-x^2}$ on \mathbb{R} . According to mathworld.wolfram.com, the system of polynomials \tilde{H}_n defined by

$$\int_{-\infty}^{\infty} \tilde{H}_m \tilde{H}_n w(x) dx = \delta_{mn} 2^n n! \sqrt{\pi}$$

satisfies the recurrence

$$\tilde{H}_0(x) = 1 \tag{4}$$

$$\tilde{H}_{n+1}(x) = 2x\tilde{H}_n(x) - 2n\tilde{H}_{n-1}(x). \tag{5}$$

Let the orthonormal Hermite polynomials be defined by $H_n(x) = \frac{1}{2^{n/2}\sqrt{n!}\pi^{1/4}}\tilde{H}_n(x)$, so that

$$\int_{-\infty}^{\infty} H_m H_n w dx = \delta_{mn},$$

and substitute $2^{n/2}(n!)^{1/2}\pi^{1/4}H_n(x)$ for $\tilde{H}_n(x)$ in (5) to find

$$\begin{aligned} H_0(x) &= \pi^{1/4} \\ xH_n(x) &= \frac{1}{\sqrt{2}}\sqrt{n}H_{n-1}(x) + \frac{1}{\sqrt{2}}\sqrt{n+1}H_{n+1}(x). \end{aligned}$$

These equations are exactly of the form (1)-(3), so the eigenvalues of

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \sqrt{1} & & & & & \\ \sqrt{1} & 0 & \sqrt{2} & & & & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \sqrt{n-2} & 0 & \sqrt{n-1} & \\ & & & & \sqrt{n-1} & 0 & \end{bmatrix}$$

are the roots of H_n .

Exercise: Write a MATLAB function that generates this tridiagonal matrix. Use the `eig` command to compute its eigenvalues for any choice of n . Histogram the roots of H_n for $n = 50, 100, 250, 300, 500$ using the `histn` function. Compare it to the semi-circular law. Does it make a difference whether the the generalized ensemble is real or complex?

3 Laguerre polynomials

Let $w_\alpha(x) = x^\alpha e^{-x}$ on $[0, \infty)$.

Exercise: Verify that the Laguerre polynomials defined by

$$\int_0^\infty L_m^\alpha L_n^\alpha w dx = \delta_{mn}$$

satisfy the recurrence

$$L_0^\alpha(x) = \frac{1}{\sqrt{\Gamma(\alpha+1)}}$$

$$xL_n^\alpha(x) = -\sqrt{n}\sqrt{\alpha+n}L_{n-1}^\alpha(x) + (\alpha+1+2n)L_n^\alpha(x) - \sqrt{n+1}\sqrt{\alpha+n+1}L_{n+1}^\alpha(x).$$

Therefore, the roots of L_n^α are given by the eigenvalues of

$$\begin{bmatrix} \alpha+1 & -\sqrt{1}\sqrt{\alpha+1} & & & & \\ -\sqrt{1}\sqrt{\alpha+1} & \alpha+3 & -\sqrt{2}\sqrt{\alpha+2} & & & \\ 0 & -\sqrt{2}\sqrt{\alpha+2} & \alpha+5 & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & -\sqrt{n-1}\sqrt{\alpha+n-1} & \alpha-1+2n \end{bmatrix}.$$

Exercise: Write a MATLAB function that generates this tridiagonal matrix. Use the `eig` command to compute its eigenvalues for any choice of n and α . Histogram the roots of L_n for different values of n and m using the `histn` function. Compare it to the Marčenko-Pastur distribution for the generalized Laguerre Ensemble. Does it make a difference whether the the generalized ensemble is real or complex? Note: $\alpha = \beta(n-m+1)/2 - 1$ where $\beta = 1$ when the elements are real and $\beta = 2$ when the elements are complex. The parameters m and n are associated with the rows and columns of the Gaussian matrix used to construct the Wishart matrix. Recall that $m/n = c$.

4 Jacobi polynomials

Jacobi polynomials are orthogonal with respect to the weight $w(x) = (1-x)^\alpha(1+x)^\beta$ on the interval $[-1, 1]$.

Exercise: Verify that the tridiagonal matrix describing the recurrence is produced by the following MATLAB code.

```

function t = trijacobi(n,a,b)
%TRIJACOBI Returns the tridiagonal matrix of polynomial recurrence coefficients
%   TRIJACOBI(N,A,B) returns a tridiagonal matrix whose entries correspond
%   to the coefficients describing the recurrence between
%   the Jacobi Polynomials.
%
%   The eigenvalues of this matrix correspond to the roots of the N-th
%   Jacobi polynomial with parameters A and B.
%
%   N, A and B are integers.
%
%
%   References:
%   [1] Alan Edelman, Handout 6: Tridiagonal Matrices, Orthogonal
%       Polynomials ans the Classical Random Matrix
%       Ensembles, Fall 2004,
%       Course Notes 18.338.
%   [2] Alan Edelman, Random Matrix Eigenvalues.
%   [3] Gabor Szego, Orthogonal Polynomials, American Mathematical
%       Society, Providence, 1975. 4th Edition.
%   [4] Eric Weisstein, Jacobi Polynomial." From MathWorld--
%       A Wolfram Web Resource.
%       http://mathworld.wolfram.com/JacobiPolynomial.htm
%
%   Brian Sutton, Sept. 2004.
%   $Revision: 1.0 $ $Date: 2004/09/23 11:35:18 $

i = (1:n-1)';

d1 = 2*sqrt(i.*(a+i).*(b+i).*(a+b+i)./(-1+a+b+2*i)./(1+a+b+2*i))./(a+ ...
        b+2*i);
d0 = -(a^2-b^2)./(a+b+2*(0:n-1)')./(2+a+b+2*(0:n-1)');

t = spdiags([[d1;0] d0 [0;d1]],[-1 0 1],n,n);

```

Code 1

Exercise: Use the `eig` command to compute its eigenvalues for any choice of n . Histogram the roots of T_n for different values of n , a and b using the `histn` function. Compare it to the MANOVA density for the MANOVA matrix. Hint: You might have to shift the spectrum for it to line up exactly. Does it make a difference whether the the generalized ensemble is real or complex? Note: $a = \beta(n_1 - m + 1)/2 - 1$ and $b = \beta(n_2 - m + 1)/2 - 1$ where $\beta = 1$ when the elements are real and $\beta = 2$ when the elements are complex. The parameters m , n_1 and n_2 are associated with the rows and columns of the Gaussian matrices used to construct the MANOVA matrix. Recall that $m/n_1 = c_1$ and $m/n_2 = c_2$.

References

- [1] Gábor Szegő. *Orthogonal Polynomials*. American Mathematical Society, Providence, 1975. 4th edition.
- [2] Percy Deift. *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*. American Mathematical Society, Providence, 1998.