18.318 (Spring 2006): Problem Set #5

due May 3, 2006

1. [5] Show that the only r-differential lattices are direct products of Y's and Z_j's. In particular, the only 1-differential lattices are Y and Z_1.

2. [5] Let P be an r-differential poset. Show that for all $i \geq 0$,

$$\#(Y^r)_i \leq \#P_i \leq (Z_r)_i,$$

where Y denotes Young's lattice and Z_r the Fibonacci r-differential lattice.

3. [2+] Let P be an r-differential poset, and let $\kappa(n \to n+k \to n)$ be the number of *closed* Hasse walks in P that start at rank n, go up to rank $n+k$ in k steps, and then go back down to the original starting vertex at rank n in k steps. For instance, it was shown in class that $\kappa(0 \to k \to 0) = r^k k!$. Show that for fixed $k \geq 0$,

$$\sum_{n \geq 0} \kappa(n \to n+k \to n) q^n = r^k k! (1-q)^{-k} F(P,q),$$

where $F(P,q)$ denotes the rank-generating function of P.

Hint. Begin with

$$\kappa(n \to n+k \to n) = \sum_{x \in P_n} \langle D^k U^k x, x \rangle.$$

4. [2] Let P be an r-differential poset. Find the eigenvalues and eigenvectors of the linear transformation $DU : \mathbb{Q}P_n \to \mathbb{Q}P_n$.

5. [2–] Let U and D be linear transformations on some vector space such that $DU - UD = rI$. A linear transformation such as $UUDUDD$ which is a product of U's and D's is called *balanced* if it contains the same number of U's as D's. Show that any two balanced linear transformations commute.

6. [2+] Let P be an r-differential poset, and let $\kappa_{2k}(n)$ denote the total number of closed Hasse walks of length $2k$ starting at P_n. Show that for fixed $k \geq 0$,

$$\sum_{n\geq 0} \kappa_{2n} q^n = \frac{(2k)!r^k}{2^k k!}\left(\frac{1+q}{1-q}\right)^k F(P,q).$$

7. (a) [2+] Let P be an r-differential poset. Let $\mathcal{H}(P_[i,j])$ denote the Hasse diagram of P restricted to $P_i \cup P_{i+1} \cup \cdots \cup P_j$, considered as an (undirected) graph. Let $\operatorname{Ch}\mathcal{H}(P_[i,j]) = \det(xI - A)$, the (monic) characteristic polynomial of the adjacency matrix A of $\mathcal{H}(P_[i,j])$. Show that

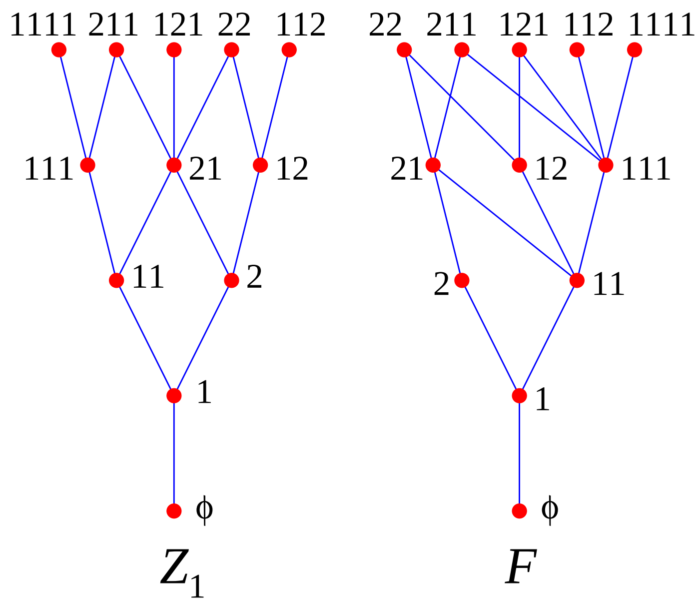
$$\operatorname{Ch}\mathcal{H}(P_[j-2,j]) = x^{\Delta p_j}(x^2-r)^{\Delta p_{j-1}} \prod_{s=2}^{j} (x^3 - r(2s-1)x)^{\Delta p_{j-s}},$$

where $p_i = \#P_i$ and $\Delta p_i = p_i - p_{i-1}$.

(b) [3–] Generalize to $\operatorname{Ch}\mathcal{H}(P_[j-k,j])$ for any $k \geq 0$. Express your answer in terms of the characteristic polynomials of certain matrices depending only on j, k and r, none larger than $(k+1)\times(k+1)$.

8. The elements x of Z_1 can be labelled in a simple way by sequences $\alpha(x)$ of 1's and 2's, so that the rank of an element labelled $a_1 \cdots a_k$ is $a_1 + \cdots + a_k$. Namely, first label the bottom element $\hat{0}$ by $\emptyset$ (the empty sequence), then the unique element covering $\hat{0}$ by 1, and then the two elements of rank 2 by 11 and 2. Now assume that we have labelled all elements up to rank n. If x has rank $n+1$, then let y be the meet of all elements that x covers. Let $k = \operatorname{rank}(x) - \operatorname{rank}(y)$. It is easy to see that $k = 1$ or $k = 2$. Define $\alpha(x) = k\alpha(y)$, i.e, preprend k to the label of y.

   There is a another poset F whose elements are also labelled by sequences of 1's and 2's, viz., order all such sequences componentwise (regarding the sequences as terminating in infinitely many 0's). For instance, $\emptyset < 1 < 2 < 21 < 211 < 212 < 2121 < 2221 < 22211$ is a saturated chain in F.



(a) [3–] Suppose that $x \in Z_1$ and $x' \in F$ have the same labels. Show that $e(x) = e(x')$, where in general $e(y)$ denotes the number of saturated chains from $\hat{0}$ to y.

(b) [3] More generally, show that for any i, the number of chains $\hat{0} < x_1 < \cdots < x_i = x$ of length i from $\hat{0}$ to x in Z_1 is the same as the number of such chains from $\hat{0}$ to x' in F.

9. [5–] Suppose that A and B are two commuting $g \times g$ nilpotent matrices. Assume that A and B are *jointly* generic, i.e., the nonzero entries of A and B together are algebraically independent over $\mathbb{Q}$. What can be said about the invariants (Jordan block sizes) of A, B, and AB, in terms of the labelled acyclic digraphs corresponding to A and B? What about the special case $AB = BA = 0$? (I don't know whether this problem has received any attention.)

10. [3–] Show that the number of $n \times n$ nilpotent matrices over $\mathbb{F}_q$ is equal to $q^{n(n-1)}$.