18.318 (Spring 2006): Problem Set #4

due April 19, 2006

1. [2] Let P_n be the set of all planted forests (i.e., graphs for which every component is a rooted tree) on the vertex set $[n]$. Let uv be an edge of a forest $F \in P_n$ such that u is closer to the root of its component than v. Define F to *cover* F' if F' is obtained from F by removing the edge uv and rooting the new tree containing v at v. This definition of cover defines the cover relation of a partial ordering of P_n. Show that P_n has the Sperner property. (HINT. Mimic Lubell's proof that B_n is Sperner.)

2. (a) [3+] Find an explicit injection $\mu : L(m,n)_i \to L(m,n)_{i+1}$ for $0 \leq i < \frac{1}{2}mn$.

   (b) [5] Find μ as in (a) such that μ is also an order-matching.

   (c) [5] We say that a graded rank-symmetric poset P of rank n has a *symmetric chain decomposition* if we can write P as a disjoint union of saturated chains C, such that each C starts at some P_i and ends at P_{n-i}. Show that $L(m,n)$ has a symmetric chain decomposition.

3. Let q be a prime power, and let V be an n-dimensional vector space over $\mathbb{F}_q$. Let $B_n(q)$ denote the poset of all subspaces of V, ordered by inclusion. It's easy to see that $B_n(q)$ is graded of rank n, the rank of a subspace of V being its dimension.

   (a) [2–] Show that the number of elements of $B_n(q)$ of rank k is given by the *q-binomial coefficient*

   $$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}.$$

   (One way to do this is to count in two ways the number of k-tuples $(v_1,\dots,v_k)$ of linearly independent elements from $\mathbb{F}_q^n$: (1) first choose v_1, then v_2, etc., and (2) first choose the subspace W spanned by $v_1,\dots,v_k$, and then choose v_1, v_2, etc.)

   (b) [1+] Show that $B_n(q)$ is rank-symmetric. (The easiest way is to use (a).)

(c) [1+] Show that every element $x \in B_n(q)_k$ covers $[k] = 1 + q + \cdots + q^{k-1}$ elements and is covered by $[n-k] = 1 + q + \cdots + q^{n-k-1}$ elements.

(d) [2–] Define operators $U_i : \mathbb{R}B_n(q)_i \to \mathbb{R}B_n(q)_{i+1}$ and $D_i : \mathbb{R}B_n(q)_i \to \mathbb{R}B_n(q)_{i-1}$ by

$$U_i(x) = \sum_{\substack{y \in B_n(q)_{i+1} \\ y > x}} y$$

$$D_i(x) = \sum_{\substack{z \in B_n(q)_{i-1} \\ z < x}} z.$$

Show that

$$D_{i+1}U_i - U_{i-1}D_i = ([n-i] - [i])I_i.$$

(e) [2–] Deduce that $B_n(q)$ is rank-unimodal and Sperner.

(f) [5] Let $0 \le i < n/2$. Find an explicit order-matching $\mu : B_n(q)_i \to B_n(q)_{i+1}$.

4. Let $M(n)$ be the set of all subsets of $[n]$, with the ordering $A \le B$ if the elements of A are $a_1 > a_2 > \cdots > a_j$ and the elements of B are $b_1 > b_2 > \cdots > b_k$, where $j \le k$ and $a_i \le b_i$ for $1 \le i \le j$. (The empty set $\emptyset$ is the bottom element of $M(n)$.)

   (a) [1+] Draw the Hasse diagrams (with vertices labelled by the subsets they represent) of $M(1)$, $M(2)$, $M(3)$, and $M(4)$. BONUS: Also do $M(5)$.

   (b) [1+] You may assume that $M(n)$ is graded of rank $\binom{n+1}{2}$, with $\rho(\{a_1, \ldots, a_j\}) = \sum a_i$. (This is very easy to prove.) Show that the rank-generating function of $M(n)$ is given by

$$F(M(n), q) = (1+q)(1+q^2)\cdots(1+q^n).$$

   (c) [2+] Define linear transformations

$$U_i : \mathbb{R}M(n)_i \to \mathbb{R}M(n)_{i+1}, \quad D_i : \mathbb{R}M(n)_i \to \mathbb{R}M(n)_{i-1}$$

by

$$\begin{aligned} U_i(x) &= \sum_{\substack{y\in M(n)_{i+1}\\ x<y}} y, \quad x \in M(n)_i \\ D_i(x) &= \sum_{\substack{v\in M(n)_{i-1}\\ v<x}} c(v,x)v, \ \ x \in M(n), \end{aligned}$$

where the coefficient $c(v, x)$ is defined as follows. Let the elements of v be $a_1 > \cdots > a_j > 0$ and the elements of x be $b_1 > \cdots > b_k > 0$. Since x covers v, there is a unique r for which $a_r = b_r - 1$ (and $a_k = b_k$ for all other k). In the case $b_r = 1$ we set $a_r = 0$. (E.g., if x is given by $5 > 4 > 1$ and v by $5 > 4$, then $r = 3$ and $a_3 = 0$.) Set

$$c(v,x) = \begin{cases} \binom{n+1}{2}, & \text{if } a_r = 0 \\ (n-a_r)(n+a_r+1), & \text{if } a_r > 0. \end{cases}$$

Show that

$$D_{i+1}U_i - U_{i-1}D_i = \left(\binom{n+1}{2} - 2i\right) I_i,$$

where I_i denotes the identity linear transformation on $\mathbb{R}M(n)_i$.

(d) [2+] Show that $M(n)$ is rank-symmetric, rank-unimodal, and Sperner. In particular, the polynomial $(1+q)(1+q^2)\cdots(1+q^n)$ has unimodal coefficients.

Hint. You may assume the following result from linear algebra. For two proofs, see pp. 331-333 of *Selected Papers on Algebra* (S. Montgomery, *et al.*, eds.), Mathematical Association of America, 1977.

Theorem. *Let V and W be finite-dimensional vector spaces over a field. Let $A : V \to W$ and $B : W \to V$ be linear transformations. Then*

$$x^{\dim V} \det(AB - xI) = x^{\dim W} \det(BA - xI).$$

In other words, AB and BA have the same nonzero eigenvalues.

(e) [2] Let S be a finite subset of $\mathbb{R}^+$ and $\alpha \in \mathbb{R}^+$. Define

$$f(S,\alpha) = \#\left\{T \subseteq S \,:\, \sum_{t\in T} t = \alpha\right\},$$

the *total* number of subsets of S whose elements sum to α. (The sum of the elements of the empty set $\emptyset$ is taken to be 0.) Show that if $S \in \binom{\mathbb{R}^+}{n}$ and $\alpha \in \mathbb{R}^+$, then

$$f(S,\alpha) \le f\left([n], \left\lfloor \frac{1}{2}\binom{n+1}{2}\right\rfloor\right).$$

(f) [1+] Let

$$h(n) = \max f(S,\alpha),$$

where the maximum is taken over all n-element subsets S of $\mathbb{R}^+$ and all $\alpha \in \mathbb{R}^+$. Show that $h(n)$ is equal to the coefficient of $q^{\lfloor \frac{1}{2}\binom{n+1}{2}\rfloor}$ in the polynomial $(1+q)(1+q^2)\cdots(1+q^n)$.

(g) [5] Find an explicit injection (or even better, an order-matching) $M(n)_i \to M(n)_{i+1}$ for $0 \le i < \frac{1}{2}\binom{n+1}{2}$.

5. [2+] Let G be a subgroup of $\mathfrak{S}_n$. Show that the rank-generating function of B_n/G is given by

$$F(B_n/G, q) = \frac{1}{\#G}\sum_{\pi\in G}\prod_C (1+q^{\#C}),$$

where C ranges over all cycles in the disjoint cycle decomposition of π. For instance, if $\pi = (1,6,9,2)(3)(4,7)(5,8)$, then

$$\prod_C (1+q^{\#C}) = (1+q)(1+q^2)^2(1+q^4).$$

Hint. Use Burnside's lemma, which states that if H is any group of permutations of a finite set, then the number of orbits of H is equal to the average number of fixed points of elements of H.

6. (a) [3–] Let $P = P_0 \cup P_1 \cup \cdots \cup P_n$ be a finite rank-symmetric, rank-unimodal poset that is graded of rank n. Let $p_i = \#P_i$ as usual. Show that the following three conditions are equivalent:

(i) For all $k \geq 1$, the largest union of k antichains of P is obtained by taking the union of the largest k ranks (or levels) of P.

(ii) For all $0 \leq i < n/2$, there exist p_i pairwise disjoint saturated chains from P_i to P_{n-i} .

(iii) For $0 \leq i < n$ there exist order-raising operators $\varphi_i : \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ such that for all $0 \leq j < n/2$ the composite linear transformation

$$\varphi_{n-j-1}\varphi_{n-j-2}\cdots\varphi_{j+1}\varphi_j : \mathbb{R}P_j \to \mathbb{R}P_{n-j}$$

is a bijection.

HINT. For (iii)⇒(ii) use the Cauchy-Binet theorem.

(b) [2+] Show that the above conditions hold for the quotient posets B_n/G, where G is a subgroup of $\mathfrak{S}_n$.

(c) [2] Give an example of a poset satisfying the conditions (i)-(iii) above that does not have a symmetric chain decomposition (as defined in Exercise 2(c)).

7. (a) [3+] Let $L(n_1, \ldots, n_k)$ denote the lattice of order ideals of $\boldsymbol{n_1} \times \cdots \times \boldsymbol{n_k}$, where $\boldsymbol{n}$ denotes an n-element chain (agreeing with our previous notation $L(m,n)$). It is easy to see that $L(n_1, \ldots, n_k)$ is rank-symmetric. By constructing suitable order-raising and lowering operators show that $L(m,n,k)$ is rank-unimodal and Sperner.

   (b) [5] Show that $L(m,n,k,h)$ is rank-unimodal and Sperner.

8. [2+] Show that the polynomials

$$\begin{bmatrix} 2n \\ n \end{bmatrix} \pm (1+q)(1+q^3)(1+q^5)\cdots(1+q^{2n-1})$$

have unimodal coefficients.

HINT. We constructed order-raising operators

$$U_i : \mathbb{R}L(n,n)_i \to \mathbb{R}L(n,n)_{i+1}$$

as "quotients" of those on $(B_{n^2})_i$. Consider in addition the nontrivial automorphism of $L(n,n)$.

9. (a) [3+] Let A denote the adjacency matrix of the Hasse diagram of $L(m,n)$ (considered as a graph). In other words, the rows and columns of A are indexed by the elements of $L(m,n)$, and

$$A_{\lambda\mu} = \begin{cases} 1, & \text{if } \lambda \text{ covers } \mu \text{ or } \mu \text{ covers } \lambda \text{ in } L(m,n), \\ 0, & \text{otherwise.} \end{cases}$$

Show that when the characteristic polynomial $\det(xA - I)$ is factored over $\mathbb{Q}$, the degree of every irreducible factor divides

$$\frac{1}{2}\phi(2(m+n+1)),$$

where ϕ denotes the Euler phi-function.

(b) [5] For what m and n is A invertible? More strongly, find the rank of A.

10. [5] Investigate when B_n/G is a lattice or distributive lattice. E.g., if B_n/G is a distributive lattice, then is it isomorphic to a product of $L(i,j)$'s? Are there any examples of B_n/G being a nondistributive lattice besides the example given in class when $n = 6$?