18.318 (Spring 2006): Problem Set #3

due April 5, 2006

1. A simplicial complex Δ is *2-acyclic* (over the field K) if Δ is acyclic and $\mathrm{lk}(v)$ is acyclic for every vertex v of Δ. Here lk denotes link, and "acyclic" means that all reduced homology vanishes over K.

   (a) [3+] Let K be a field. Show that $(k_0, \ldots, k_{d-1})$ is the f-vector of some 2-acyclic (over K) simplicial complex Δ of dimension $d-1$ if and only if there exists a simplicial complex Γ of dimension $d-3$ such that
$$\sum_{i\geq 0} k_{i-1}x^i = (1+x)^2 \sum_{i\geq 0} f_{i-1}(\Gamma)x^i.$$

   (b) [5] Show that if Δ is 2-acyclic (over any field K, or possibly over all fields K or over $\mathbb{Z}$), then the face poset of Δ can be partitioned into a disjoint union of boolean algebras of rank 2.

2. [3–] Let Δ be a simplicial complex of dimension $d-1$ with n vertices. Show that if $d \leq n/2$ then
$$|\tilde{\chi}(\Delta)| \leq \binom{n-1}{d},$$
and that this inequality is best possible.

3. [2] Let Δ be a $(d-1)$-dimension Eulerian simplicial complex, *except* that we put no restriction on $\tilde{\chi}(\Delta)$. (We then say that Δ is *semi-Eulerian.*) Let $h_i = h_i(\Delta)$. Determine the polynomial $P(x)$ such that
$$\sum h_i x^i = \sum h_{d-i}x^i + P(x).$$
(If Δ is Eulerian, i.e., if we add the additional condition that $\tilde{\chi}(\Delta) = (-1)^{d-1}$, then $P(x) = 0$ by the Dehn-Sommerville equations.)

4. [2+] Let $F(q)$ and $G(q)$ be symmetric unimodal polynomials with non-negative real coefficients. Show that $F(q)G(q)$ is also symmetric (easy) and unimodal (harder).

5. [2+] A polynomial $a_0 + a_1 x + \cdots + a_n x^n$ with positive real coefficients is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for $1 \leq i \leq n-1$. Show that if $F(x)$ and $G(x)$ are log-concave, then so is $F(x)G(x)$.

   HINT. Let $F(x) = \sum a_i x^i$. Consider the infinite matrix $A_F = (a_{j-i})_{i,j\geq 0}$ and similarly A_G. (Set $a_j = 0$ if $j < 0$ or $j > \deg F$.) Apply the Cauchy-Binet theorem to certain minors of $A_F A_G$.

6. [2–] A $(0,1)$-*necklace* of *length* n and *weight* i is a circular arrangement of i 1's and $n - i$ 0's. For instance, the $(0,1)$-necklaces of length 6 and weight 3 are (writing a circular arrangement linearly) 000111, 001011, 010011, and 010101. (Cyclic shifts of a linear word represent the same necklace, e.g., 000111 is the same as 110001.) Let N_n denote the set of all $(0,1)$-necklaces of length n. Define a partial order on N_n by letting $u \leq v$ if we can obtain v from u by changing some 0's to 1's. It's easy to see (you may assume it) that N_n is graded of rank n, with the rank of a necklace being its weight. Show that N_n is rank-symmetric, rank-unimodal, and Sperner.

7. [3] Is the f-vector of a pure simplicial complex unimodal?