

18.318 (Spring 2006): Problem Set #2

due March 8, 2006

- [2+] Let G be a finite abelian group of order n , written additively. An $n \times n$ matrix $A = (a_{uv})$ over a field K whose rows and columns are indexed by G is said to have G -symmetry if there is a function $f : G \rightarrow K$ such that $a_{uv} = f(u - v)$ for all $u, v \in G$. Fix $j \geq 1$. Prove that there exist only finitely many Hadamard matrices with the symmetry of a direct product of j cyclic 2-groups. (A 2-group is a group whose order is a power of 2.) You only need to explain how the proof for G cyclic (the case $j = 1$) given in class needs to be modified; you don't have to write down every detail of the proof. (The relevant theorem on the eigenvalues of matrices with the symmetry of an abelian group is given e.g. in EC2, Exercise 5.68.)
- [1+] Let Δ be a simplicial complex with 159 3-dimensional faces. For $i \geq 2$, find the smallest possible number of i -faces of Δ . For $i \geq 4$, find the largest possible number of i -faces of Δ . (An i -face is a face of dimension i .)
- A $(d - 1)$ -dimensional simplicial complex Δ is *pure* if all its facets (= maximal faces) have dimension $d - 1$. Let $(f_0, f_1, \dots, f_{d-1})$ be the f -vector of a pure $(d - 1)$ -dimensional simplicial complex.
 - [2+] Show that the vector $(f_{d-2}, f_{d-3}, \dots, f_0, 1)$ is the f -vector of a simplicial complex.
 - [3-] Show that $f_i \leq f_{d-2-i}$ for $-1 \leq i \leq \lfloor (d - 3)/2 \rfloor$, and that $f_0 \leq f_1 \leq f_2 \leq \dots \leq f_{\lfloor (d-1)/2 \rfloor}$.
 - [5+] Characterize f -vectors of pure simplicial complexes. (This is considered hopeless.)
- [3-] Give an example of a pure simplicial complex Δ of dimension $d - 1$ and with n vertices that fails for some i to satisfy the "Upper Bound Inequality" $h_i \leq \binom{n-d+i-1}{i}$.
- [2+] Give an example of two simplicial complexes Δ_1 and Δ_2 such that $|\Delta_1| \approx |\Delta_2|$ but such that $h_i(\Delta_1) \geq 0$ for all i and $h_j(\Delta_2) < 0$ for some

j. What is the smallest possible dimension of Δ_1 and Δ_2 ? (The symbol \approx denotes homeomorphism.)

6. Let Γ and Δ be simplicial complexes on disjoint vertex sets V and W , respectively. Define the *join* $\Gamma * \Delta$ to be the simplicial complex on the vertex set $V \cup W$ with faces $F \cup G$, where $F \in \Gamma$ and $G \in \Delta$. (If Γ consists of a single point, then $\Gamma * \Delta$ is the *cone* over Δ . If Γ consists of two disjoint points, then $\Gamma * \Delta$ is the *suspension* of Δ .)

(a) [2] Compute the h -vector $h(\Gamma * \Delta)$ in terms of $h(\Gamma)$ and $h(\Delta)$. (It is easiest to state the answer in terms of h -polynomials, where the h -polynomial of a simplicial complex Λ is $h(\Lambda, t) = \sum_i h_i(\Lambda)t^i$.)

(b) [2-] The *boundary* $\Delta(d)$ of the $(d-1)$ -dimensional cross-polytope (as an abstract simplicial complex) has vertex set

$$V = \{x_1, \dots, x_d, y_1, \dots, y_d\},$$

with $F \subseteq V$ a face of $\Delta(d)$ if and only if $\{x_i, y_i\} \not\subseteq F$ for all i . Find the h -vector $h(\Delta(d))$.

(c) [2+] Show that if Γ and Δ are Cohen-Macaulay simplicial complexes, then so is their join $\Gamma * \Delta$.

7. [2] Let Δ be a $(d-1)$ -dimensional simplicial complex. For $0 \leq j \leq d-1$, define the j -skeleton Δ_j of Δ by $\Delta_j = \{F \in \Delta : \dim F \leq j\}$. Express the h -vector $h(\Delta_{d-2})$ in terms of (the entries of) $h(\Delta)$.

8. [2+] A *matroid complex* is a simplicial complex Δ on a vertex set V such that the restriction Δ_W of Δ to any subset $W \subseteq V$ is pure. Show that a matroid complex is Cohen-Macaulay. (Only use results actually proved in class.)

9. [5] Let (h_0, \dots, h_d) be the h -vector of a matroid complex. Does there exist a *pure* multicomplex Γ (i.e., all maximal faces of Γ have the same cardinality) such that Γ has exactly h_i faces with i elements, $1 \leq i \leq d$?

10. [5] Does every partial shelling of a matroid complex extend to a complete shelling? (Even the case where the facets of Δ consist of all d -subsets of an n -set is open.) The answer is believed to be “no.”

11. [2-] Find explicitly every simplicial complex Δ with the property that *every* ordering of its facets is a shelling.

12. [2] Obtain an NZD-wart decomposition of the following three face rings:

$$R = K[x, y, z]/(xz, yz), \quad R = K[x, y, z, w]/(xz, xw, yw),$$

$$R = K[x, y, z, w]/(xw, yw, zw, xyz),$$

and use it to compute the Hilbert series $F(R, t)$.

13. [2+] Let Δ be a disconnected simplicial complex. Let θ be an NZD (assumed to be homogeneous of positive degree) in $k[\Delta]$. Show that $K[\Delta]/(\theta)$ has no NZD's (of positive degree). In other words,

$$\text{depth}(K[\Delta]) = 1.$$

14. [2+] A graded algebra $K[x_1, \dots, x_n]/I$ is said to be a *complete intersection* if the ideal I is generated by a regular sequence. Find all simplicial complexes for which the face ring $K[\Delta]$ is a complete intersection.

15. [3–] A pure simplicial complex Δ of dimension $d - 1$ is *balanced* if there is a coloring of the vertices $V(\Delta)$ with d colors such that no two vertices on the same face (equivalently, forming an edge) have the same color. Suppose that Δ is balanced and Cohen-Macaulay with h -vector (h_0, \dots, h_d) . Show that (h_1, \dots, h_d) is the f -vector of a simplicial complex.

HINT. Let the colors be $1, 2, \dots, d$, and define θ_i to be the sum of all vertices colored i . Show that $\theta_1, \dots, \theta_d$ is an h.s.o.p.

16. [3–] Let Δ be a $(d-1)$ -dimensional Cohen-Macaulay simplicial complex with the action of a free involution σ . This means that $\sigma : \Delta \rightarrow \Delta$, $\sigma^2 = 1$ (so in particular σ is a bijection), and $\{x, \sigma(x)\}$ is not a face for any vertex x of Δ . (in particular, $\sigma(x) \neq x$.) Show that $h_i(\Delta) \geq \binom{d}{i}$, so $f_{d-1}(\Delta) = \sum h_i \geq 2^d$.

HINT. The action of σ on Δ extends to an action on $K[\Delta]$. Find a “nice” h.s.o.p. $\theta_1, \dots, \theta_d$ for $K[\Delta]$ such that the space

$$W = \text{span}_K\{\theta_1, \dots, \theta_d\}$$

is σ -invariant (i.e., $\sigma W = W$). Hence σ acts on $K[\Delta]/(\theta_1, \dots, \theta_d)$.