18.318 (Spring 2006): Problem Set #1

due February 22, 2006

Hand in your "best" three problems from those below. "Reasonable" collaboration is permitted, but you should not just copy someone else's solution or look up a solution from an outside source.

We will (subjectively) indicate the difficulty level of each problem as follows:

[1]	easy
[2]	moderately difficult
[3]	difficult
[4]	horrendously difficult
[5]	unsolved

Further gradations are indicated by + and –; e.g., [3–] is a little easier than [3]. A rating of [5–] denotes an unsolved problem that has received little (if any) attention and may not be difficult.

1. [2+] Suppose that n people live in Reverse Oddtown. Every club contains an even number of persons, and any two clubs share an odd number of persons. Show that the maximum number of clubs is n if n is odd and is $n-1$ if n is even.

2. [3–] Suppose that n people live in Eventown. Every club contains an even number of persons, every two clubs share an even number of persons, and no two clubs have identical membership. Show that the maximum number of clubs is $2^{\lfloor n/2 \rfloor}$.

3. [3–] Suppose that fewer than $2^{\lfloor n/2 \rfloor}$ clubs have been formed using the Eventown rules of the previous exercise. Show that another club can be formed without breaking the rules.

4. [2+] Let the edge set $E(K_n)$ of the complete graph K_n be a union of the edge sets of complete bipartite graphs $B_1, \dots, B_m$ such that every edge of K_n is covered an odd number of times. Show that $m \geq (n-1)/2$. (The minimum value of m is not known.)

5. [2+] A *complete tripartite graph* with vertex tripartition (X_1, X_2, X_3) is the graph on the disjoint vertex sets X_i with an edge between any two

vertices not in the same set X_i. Thus if $\#X_i = p_i$ then the complete tripartite graph has $p_1p_2 + p_1p_3 + p_2p_3$ edges. What is the minimum value of m?

6. [2+] Let $A_1, \dots, A_n$ be distinct subsets of an n-set X. Give a linear algebra proof that for some $x \in X$, all the sets $A_i - x$ (short for $A_i - \{x\}$) are distinct. (There is a fairly simple combinatorial proof, but that is not what is asked for.)

   HINT. Consider the incidence matrix M of the sets and their elements. Consider two cases: $\det M = 0$ and $\det M \neq 0$.

7. (a) [1+] Fix $n \geq 1$. Let S_k denote the set of all k-subsets of $[n]$, and let $\mathbb{Q}S_k$ denote the $\mathbb{Q}$-vector space with basis S_k. For $0 \leq k < n$ define a linear transformation $U_k : \mathbb{Q}S_k \to \mathbb{Q}S_{k+1}$ by

   $$U_k(x) = \sum_{\substack{y \in S_{k+1} \\ y \supset x}} y,$$

   for $x \in S_k$. Similarly for $0 < k \leq n$ define $D_k : \mathbb{Q}S_k \to \mathbb{Q}S_{k-1}$ by

   $$D_k(x) = \sum_{\substack{y \in S_{ki1} \\ y \subset x}} y,$$

   for $x \in S_k$. Show that (multiplying linear transformations right-to-left)

   $$D_{k+1}U_k - U_{k-1}D_k = (n - 2k)I_k,$$

   where I_k is the identity transformation on $\mathbb{Q}S_k$.

   (b) [2] Deduce from (a) that U_k is injective for $k < n/2$ and surjective for $k \geq n/2$.

   (c) [3–] Let $k < n/2$. Let M_k be the matrix of the linear transformation $U^{n-2k} : \mathbb{Q}S_k \to \mathbb{Q}S_{n-k}$ with respect to the bases S_k and S_{n-k}, where U^{n-2k} is short for $U_{n-k-1}U_{n-k-2}\cdots U_k$. Find the determinant of M_k, up to sign.

8. [5] The capacity $\Theta(C_7)$ of a 7-cycle (heptagon) is not known. In 2002 it was shown that $\Theta(C_7) \geq 108^{1/4} = 3.2237\cdots$. An upper bound is clearly 7/2; I don't know whether this has been improved.