HOMEWORK 8 (18.315, FALL 2005)

1) Give a direct combinatorial proof of the hook-length formula for the number of standard Young tableaux of shape $(n-2k, 2, \ldots, 2)$, k times.

2) Fix a tree t with n vertices and root R. Let $S(v)$ denotes the shortest path from v to R. Let $\beta(v)$ be the number of vertices v' such that $S(v) \subseteq S(v')$. E.g. $\beta(R) = n$, and $\beta(v) = 1$ for every leaf $v \neq R$. Denote by $A(t)$ the set of bijections $\gamma : t \to [n]$ such that: $\gamma(v) < \gamma(v')$ for all vertices $v, v' \in t$ with $S(v) \subseteq S(v')$; e.g. this implies that $\gamma(R) = 1$. These bijections are called *increasing trees* of shape t. Prove that

$$|A(t)| = \frac{n!}{\prod_{v \in t} \beta(v)}.$$

3) Let Q be the set of partitions whose Young diagram is tileable by dominoes. Compute the generating function $\sum_{\lambda \in Q} t^{|\lambda|}$.

4) Let $\Gamma(\lambda)$ be a graph on $\mathrm{SYT}(\lambda)$, where two tableaux are connected by an edge if they differ at exactly two places. Prove that $\Gamma(\lambda)$ is connected. Prove the analogue of this result for increasing trees of the same shape.

5) Let $p_1, \ldots, p_n$ be probabilities to be born on days $1, \ldots, n$ of the year (usually $n = 365$). Let A be the event that k people (chosen independently) are all born on different days. Prove that $\mathbf{Pr}(A)$ maximizes when $p_i = 1/n$.

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