HOMEWORK 7 (18.315, FALL 2005)

1) Compute the number of permutations $\sigma \in S_n$ which are

a) both (123)- and (321)-avoiding

b) both (213)- and (312)-avoiding

c) both (123)- and (132)-avoiding

d) both (213)- and (231)-avoiding

e) both (213)- and (132)-avoiding

2) Let $\Sigma(k,\ell)$ be the set of permutations which are sortable with k queues and ℓ stacks. Prove or disprove: there exists a finite set of permutations $X(k,\ell)$ such that $\sigma \in S_n \cap \Sigma(k,\ell)$ if and only if σ avoids all permutations in X.

3) Consider only simple graphs (without loops or multiple edges). For a finite graph $G = (V, E)$, let $\alpha(G) = |E|/\binom{|V|}{2}$. For an infinite graph G let

$$\alpha_n(G) = \max_H \alpha(H)\,,$$

where the max is taken over all finite induced subgraphs of G with n vertices. Finally let $\alpha(G) = \limsup \alpha_n(G)$ as $n \to \infty$.

a) Prove that every rational number $a \in [0, 1]$ is equal to $\alpha(G)$ for some finite G.

b) Prove that for infinite graphs $\alpha(G) \neq \frac{1}{3}$.

c) Prove that for infinite graphs $\alpha(G) \in \mathbb{Q}$.

4) Let P be a polygon. A billiard trajectory in P is a (possibly self-intersecting) polygon inscribed into P such that the angle of reflection equals the angle of incidence (see Figure 1). The polygons and trajectories can have finite or infinite number of intervals.

a) Prove that every acute triangle has a finite closed billiard trajectory. (*Hint*: search for those of length 3.)

b) Let P be an acute cone. Prove that every billiard trajectory has a finite number of intervals (some of them going to infinity).

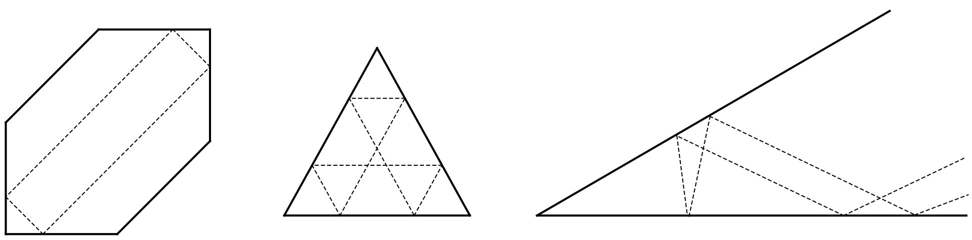

FIGURE 1. Three billiard trajectories.

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