HOMEWORK 6 (18.315, FALL 2005)

1) Let P_n be a polytope of all $n \times n$ nonnegative real matrices with row and column sums 1.

a) Prove that vertices of P_n correspond to permutations $\sigma \in S_n$.

b) Prove that the edges correspond to permutations which differ by a cycle, i.e. (v_σ, v_ω) is an edge if and only if $\omega^{-1}\sigma$ is a cycle in S_n.

c) Let Γ_n be the graph of P_n described in b). Conclude from b) that diameter of the graph of P_n is two.

d) Prove that Γ_n contains a Hamiltonian cycle.

2) Stanley, EC1, Ex. 2.16 (on Vandermonde det.)

3) Stanley, EC2, Ex. 7.66, part a) only.

4) Imagine all vertices of a graph G are drawn on a line L, and L lies in planes $P_1, \ldots, P_k \subset \mathbb{R}^3$. The edges between vertices are drawn in planes P_i without intersections, and only on one side of L. Let $c(G)$ be the smallest number K of planes needed for such drawing.

For example, $c(K_3) = 1$ since it can be embedded into one plane with all vertices along the line. Similarly, $c(K_4) = 2$ since all but one edge can be embedded into P_1, and the last edge will go into P_2.

a) Prove that if G is planar and contains a Hamiltonian cycle, then $c(G) \leq 2$.

b) Prove that if G is not planar, then $c(G) \geq 3$.

c) Prove that $c(K_n) = n/2 + O(1)$.

d) Prove that $c_n(H_n) = \theta(\log n)$, where H_n is a graph of a n-dim. hypercube.

e) Prove that if G is planar, then $c(G) \leq 1000$.
[*Hint:* use a) and Whitney thm.]

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