HOMEWORK 1 (18.315, FALL 2005)

Def. A *proper coloring* of a graph is a coloring of vertices with no monochromatic edges.
A *grid graph* $G_{m,n}$ is a product of a m-path and a n-path.

1) Let $c(n)$ be the number of proper colorings of $G_{n,n}$ with 3 colors. Prove
  a) $c(n) > C\,(1+\varepsilon)^{n^2}$ for some $C, \varepsilon > 0$;
  b) $\frac{\log c(n)}{n^2} \to \alpha$ as $n \to \infty$, for some $\alpha > 0$.

2) Denote by N_k be the number of proper colorings of $G_{n,n}$ with k colors. Approximate N_k up to 10% when
  a) $n = 100$ and $k = 1,000,000$;
  b) $n = 100$ and $k = 1,000$.

3) Consider the set $\mathcal{S}_k(n)$ of proper colorings of $G_{n,n}$ with k colors. Prove that for every two colorings $\chi, \chi' \in \mathcal{S}_k(n)$, one can go from χ to χ' by changing one color at a time, when
  a) $k = 5$;
  b) $k = 4$.

4) In Schur's theorem, the proof we presented gives $n(r) < e\,r!$. Find an exponential lower bound by an explicit construction.

5) Consider random graphs H on n vertices with $m = 2n$ edges (defined as subgraphs of a complete graph K_n). What is more likely: that H is bipartite or not?

6) An *acute decomposition* of a polygon $P \subset \mathbb{R}^2$ is a subdivision of P into acute triangles, such that there are no vertices lying on the interior edges (see Figure 1). Prove that an acute decomposition of P always exist if
  a) P is a triangle;
  b) P is a convex polygon which has an inscribed circle;
  c) P is any convex polygon.

FIGURE 1. A valid acute decomposition of a square, and an invalid one.

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Please remember to write the name(s) of your collaborators (see collaboration policy).