

Practice Set 2

This set is for your own practice; your solutions will not be collected nor graded.

9. Solve Schlömilch's integral equation,

$$f(x) = \frac{2}{\pi} \int_0^x dy \frac{u(y)}{\sqrt{x^2 - y^2}},$$

by at least two different methods. **Hint:** One way to proceed is to cast this equation into the usual Abel's equation by introducing the variables $X = x^2$ and $Y = y^2$, and then use the Laplace transform. I would propose to have some fun by applying an alternate route. You may make the change of variable $y = x \sin \theta$ under the integral sign. Then expand both $f(x)$ and $u(x \sin \theta)$ in Maclaurin series in their arguments, and integrate the right-hand side term by term by using $\int_0^{\pi/2} d\theta (\sin \theta)^n = (\sqrt{\pi}/2) \Gamma(\frac{n+1}{2})/\Gamma(\frac{n}{2} + 1)$, where $\Gamma(z)$ is the Gamma function, $\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t}$, $\text{Re } z > 0$. Thus, find the n th derivative $u^{(n)}(0)$ in terms of $f^{(n)}(0)$. Sum up the Maclaurin expansion for $u(x)$ to get $u(x)$ in terms of an integral involving the derivative of $f(x)$. Does the final formula require that $f(x)$ is infinitely differentiable? Compare your result with what you get by applying the Laplace transform to the corresponding Abel's equation mentioned above.

10. (Prob. 4.15 in text by M. Masujima.) Show that the non-trivial solutions of the equation

$$u(x) = \lambda \int_{-\pi}^{\pi} dy \left[\frac{1}{4\pi} (x-y)^2 - \frac{1}{2} |x-y| \right] u(y)$$

are $\cos(mx)$ and $\sin(mx)$, where $\lambda = m^2$ and m is any integer.

11. (Example 8.2 in text by M. Masujima.) Solve the nonlinear integral equation

$$u(x) = 1 + \lambda \int_0^x dy u(x-y) u(y).$$

In particular, show that $u(x)$ is the confluent hypergeometric function $\Phi(\frac{1}{2}; 2; 4\lambda x)$ defined as

$$\Phi(a; c; z) = 1 + \frac{a}{c} \frac{z}{1!} + \frac{a(a+1)}{c(c+1)} \frac{z^2}{2!} + \dots + \frac{a(a+1) \dots (a+n-1)}{c(c+1) \dots (c+n-1)} \frac{z^n}{n!} + \dots$$

12. (a) Show that

$$D'(\lambda) = - \int_a^b dx N(x, x; \lambda),$$

where the D and N functions were defined in class.

- (b) Consider the iterated kernel K_2 given by

$$K_2(x, y) = \int_a^b dx' K(x, x') K(x', y).$$

Show that if μ is an eigenvalue of K_2 , then either $\sqrt{\mu}$ or $-\sqrt{\mu}$ must be an eigenvalue of K .
Hint: Use the iteration series for the resolvent kernel in order to express the sum $H(x, x'; \lambda) + H(x, x'; -\lambda)$.

13. (Prob. 4.9 in text by M. Masujima.) Consider the homogeneous integral equation

$$u(x) = \lambda \int_{-\infty}^{\infty} dy K(x, y) u(y), \quad -\infty < x < \infty,$$

where t is fixed, $0 < t < 1$, and

$$K(x, y) = \frac{1}{\sqrt{1-t^2}} e^{\frac{x^2+y^2}{2}} e^{-\frac{x^2+y^2-2xyt}{1-t^2}}.$$

(a) Show directly that $u_0(x) = e^{-x^2/2}$ is an eigenfunction of K corresponding to eigenvalue $\lambda = \lambda_0 = \pi^{-1/2}$.

(b) Let $u_n(x) = e^{-x^2/2} H_n(x)$, where $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ are Hermite polynomials ($n = 0, 1, 2, \dots$). Assume that $u_n = \lambda_n K u_n$ and show that $u_{n+1} = \lambda_{n+1} K u_{n+1}$, where $\lambda_n = t \lambda_{n+1}$. This means that the original integral equation has eigenvalues $\lambda_n = t^{-n} \pi^{-1/2}$, with corresponding eigenfunctions $u_n(x)$.

14. (Prob. 4.4, Chap. 6 in text by I. Stakgold.) Heat conduction in a homogeneous infinite rod without sources or sinks can be described by the equation

$$U(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} dy e^{-\frac{(x-y)^2}{4t}} u(y), \quad -\infty < x < \infty,$$

where $U(x, t)$ is the temperature at time $t > 0$ and $u(x) = U(x, 0)$ is the temperature distribution at $t = 0$. Suppose that one seeks to determine $u(x)$ by measuring $U(x, t)$ at some given time $t = t_0 > 0$, say at $t_0 = 1$. Then the above equation can be viewed as an integral equation for $u(x)$ with given $U(x, 1)$, which is a case of a Fredholm integral equation of the first kind. Discuss the possibility of solving this integral equation. In particular, show that a solution is possible if $U(x, 1)$ is *band limited*, i.e., its Fourier transform vanishes outside some finite interval. **Remark:** For some people, this problem is considered as a typical example of an *ill-posed* problem in mathematical physics.

15. (Prob. 5.14 in text by M. Masujima.) If $|h| < 1$, find the non-trivial periodic solutions of the integral equation

$$u(x) = \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} dy \frac{1-h^2}{1-2h \cos(x-y) + h^2} u(y).$$

Evaluate the corresponding values for the parameter λ .

Hint: The kernel of this equation is translationally invariant. Write $\cos(x-y)$ as a sum of two exponentials, express the kernel in terms of the complex variable $\zeta = e^{i(y-x)}$, use partial fractions, and then expand each fraction in powers of ζ .

16. (Similar to Prob. 5.2 in text by M. Masujima.) Consider the differential equation

$$\frac{d}{dx} \left(x \frac{du}{dx} \right) = -\lambda xu, \quad 0 < x < 1,$$

with the boundary conditions $u(0)$: finite, and $u(1) = 0$; λ is a constant parameter.

(a) Convert this Sturm-Liouville problem into a homogeneous Fredholm integral equation. Symmetrize the kernel of this equation “as much as possible.”.

(b) Assuming that the given Sturm-Liouville problem has discrete, positive eigenvalues $\lambda = \lambda_n$, $n = 0, 1, 2, \dots$, give an estimate of the lowest eigenvalue λ_0 ($0 < \lambda_0 < \lambda_1 < \dots < \lambda_n < \dots$).

17. (Read Examples 4.3 & 4.4 in text by M. Masujima.) Solve the integral equation

$$u(x) = e^{-a|x|} + \lambda \int_{-\infty}^{\infty} dy e^{-|x-y|} u(y), \quad -\infty < x < \infty, \quad a > 0,$$

by converting it into a 2nd-order differential equation for $x < 0$ and $x > 0$. Compare your solution with the one derived in class. **Hint:** You may seek a particular solution that is even and continuous. Also, seek a condition for $u'(0^+)$ and $u'(0^-)$ that you need to impose when you solve the differential equation.