18.307: Integral Equations **M.I.T. Department of Mathematics**

Spring 2006

Practice Set 1 **Due:** NEVER

This set is for your own practice; your solutions will not be collected nor graded.

1. (Prob. 8.4 in text by M. Masujima.) Solve the nonlinear integral equation

$$\rho(\phi) = \exp\left[\int_0^{2\pi} d\theta\, \cos(\phi-\theta)\, \rho(\theta)\right].$$

2. (Prob. 2.6 in text by M. Masujima.) Solve the integral equation

$$u(x) = 1 + \int_0^1 dy\,(1+x+y+xy)^\nu\, u(y), \quad 0 \le x \le 1, \quad \nu: \text{ real}.$$

Hint: Notice that $1+x+y+xy$ can be factorized.

3. (Prob. 2.3 in text by M. Masujima.) Consider the integral equation

$$u(x) = f(x) + \lambda \int_{-\infty}^{\infty} dy\, e^{-x^2-y^2}\, u(y), \quad -\infty < x < \infty.$$

(a) Solve this equation for $f = 0$. For what values of λ does it have non-trivial solutions?
(b) Solve the equation for $f(x) = x^m$, where $m = 0, 1, 2, \ldots$. Does this inhomogeneous equation have any solutions when λ is equal to an eigenvalue of the kernel? Explain.
Hint: Distinguish cases for m. You may wish to express your results in terms of the Gamma function, $\Gamma(z) = \int_0^\infty dt\, e^{-t}\, t^{z-1}$, Re $z > 0$.

4. (From Sec. 2.2 in text by M. Masujima.) (a) Instead of using the Fourier transform, find the Green function for the initial-value problem of the Schrödinger equation done in class by setting

$$G(x,y) = \begin{cases} A\, e^{ik(x-y)} + B\, e^{-ik(x-y)}, & x > y, \\ C\, e^{ik(x-y)} + D\, e^{-ik(x-y)}, & x < y, \end{cases}$$

and determining A, B, C and D directly from the appropriate conditions for $G(x,y)$ at $x = 0$ and $x = y$.
(b) Find the Green function for the scattering problem of the Schrödinger equation done in class, where $\psi(x) \sim e^{ikx} + R\, e^{-ikx}$ as $x \to -\infty$ and $\psi(x) \sim T\, e^{ikx}$ as $x \to +\infty$, by casting $G(x,y)$ in the above form and avoiding the Fourier transform.

5. (From Prob. 3.2 in text by M. Masujima.) This problem explores different choices of the Green function in a version of Schrödinger's equation.
(a) Convert the radial Schrödinger equation

$$\frac{d^2}{dr^2}\psi(r) - \frac{l(l+1)}{r^2}\,\psi(r) = [V(r) - k^2]\,\psi(r), \quad r > 0,$$

into an integral equation by replacing the right-hand side of this equation by $\delta(r-r')$. The wave function $\psi(r)$ satisfies the initial condition

$$\psi(r) \sim r^{l+1} \qquad \text{as } r \to 0^+.$$

Show that the solution is an analytic function of l for $\mathrm{Re}\, l \geq -\frac{1}{2}$. **Hint:** Show that the Green function for this problem is $G(r,r') = 0$ for $r < r'$, and

$$G(r,r') = (2l+1)^{-1} \left[\frac{r^{l+1}}{(r')^l} - \frac{(r')^{l+1}}{r^l} \right], \qquad r > r'.$$

(b) In part (a), define the Green function by

$$\frac{\partial^2}{\partial r^2} G(r,r') - \frac{l(l+1)}{r^2} G(r,r') + k^2\, G(r,r') = \delta(r-r'),$$

i.e., by including the contribution of k^2 in the left-hand side. Determine this $G(r,r')$ under the same initial condition for $\psi(r)$.

6. (Prob. 3.16 in text by M. Masujima.) Consider the Volterra integral equation of the 2nd kind

$$u(x) = f(x) + \lambda \int_0^x dy\, e^{x^2-y^2}\, u(y), \quad x > 0.$$

(a) Sum up the iteration (Liouville-Neumann) series *exactly* and find the general solution to this equation. Verify that the solution is analytic in λ.
(b) As a check, solve this integral equation by converting it into a differential equation. **Hint:** Multiply both sides by e^{-x^2} and differentiate.

7. (Prob. 2.13 in text by M. Masujima.) Discuss how you would solve the Volterra integral equation of the 2nd kind,

$$u(x) = f(x) + \lambda \int_0^x dy\, K(x,y)\, u(y),$$

with the kernel given by $K(x,y) = \sum_{n=1}^N g_n(x)\, h_n(y)$.

8. (Prob. 3.3 in text by M. Masujima.) (a) Discuss how you would solve the integral equation

$$\int_0^x dy\, K(x-y)\, u(y) = f(x), \quad x > 0,$$

in which the kernel depends on the difference $(x-y)$ (such kernels are called "difference kernels" in part of the literature). This equation is called a Volterra equation of the 1st kind (the u is missing outside the integral).
(b) Apply your method to the case $K(x) = x^{-\nu}$, where $0 < \nu < 1$. The resulting equation is called generalized Abel's equation.