Solutions to Set 7

(21)

$$F(\jmath) = \int_a^b dx \; e^{-i\jmath x} \; f(x), \qquad I(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\jmath \; e^{i\jmath z} \; F(\jmath).$$

We evaluate $I(z)$ for $\underline{a < z < b}$ as follows.

$$I(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\jmath \; e^{i\jmath z} \int_a^b dx' \; e^{-i\jmath x'} \; f(x').$$

In order to interchange the order of integration, we deform the path in x' to the lower z'-plane $(z' = x' + iy')$ for $\underline{\jmath > 0}$ and to the upper z'-plane for $\underline{\jmath < 0}$:

$$I(z) = \frac{1}{2\pi} \left[\int_{-\infty}^{0} d\jmath \; e^{i\jmath z} \int_{C_2} dz' \; e^{-i\jmath z'} f(z') \right.$$

z'-plane

$C_2 = -\bar{C}_2$

$a \qquad b$

C_1

$$+ \int_{0}^{\infty} d\jmath \; e^{i\jmath z} \int_{C_1} dz' \; e^{-i\jmath z'} f(z') \left. \vphantom{\int} \right]$$

$$= \frac{1}{2\pi} \left[\int_{C_2} dz' \; f(z') \int_{-\infty}^{0} d\jmath \; e^{i(z-z')\jmath} + \int_{C_1} dz' \; f(z') \int_{0}^{\infty} d\jmath \; e^{i(z-z')\jmath} \right]$$

The exponent in the first integral is $i(z-z')\jmath$ with $\jmath < 0$ and $\mathrm{Im}(z-z') < 0$; hence, $\mathrm{Re}[i(z-z')\jmath] < 0$ and the integral converges absolutely.

The exponent in the second integral is $i(z-z')\jmath$ with $\jmath > 0$ and $\mathrm{Im}(z-z') > 0$; hence, $\mathrm{Re}[i(z-z')\jmath] < 0$ and the integral converges absolutely.

$$I(z) = \frac{1}{2\pi} \left[\int_{C_2} dz'\, f(z')\, \frac{1}{i(z-z')} + \int_{C_1} dz'\, f(z')\, \frac{1}{i(z'-z)} \right]$$

$$= \frac{1}{2\pi i} \left[+ \int_{\bar{C}_2} dz'\, \frac{f(z')}{z'-z} + \int_{C_1} dz'\, \frac{f(z')}{z'-z} \right] = \frac{1}{2\pi i} \int_{C_1 \cup \bar{C}_2} dz'\, \frac{f(z')}{z'-z}$$

$$\Rightarrow \quad I(z) = \frac{1}{2\pi i} \int_{C_1 \cup \bar{C}_2} dz'\, \frac{f(z')}{z'-z} = f(z), \qquad a < z < b,$$

by the Cauchy integral theorem, since $f(z)$ is analytic inside $C_1 \cup \bar{C}_2$ and along $C_1 \cup \bar{C}_2$.

Clearly, if $\underline{z < a}$ or $\underline{z > b}$,

$$I(z) = 0.$$

(22) $\quad \int_0^\infty dy \ K_0(x-y) \cdot u(y) = 2\pi \ , \quad x \geqslant 0$

Let
$$g(x) = \begin{cases} 0, & x \geqslant 0 \\ \int_0^\infty dy \ K_0(x-y) \cdot u(y), & x < 0, \end{cases} \qquad h(x) = \begin{cases} 2\pi, & x \geqslant 0, \\ 0, & x < 0, \end{cases}$$

while $\quad u(x) = 0, \quad x < 0.$ The given IE is extended to $-\infty < x < \infty$ as

$$\int_{-\infty}^\infty dy \ K_0(x-y) \ u(y) = g(x) + h(x) \ , \qquad -\infty < x < \infty.$$

The FT of this equation gives

$$\tilde{K}_0(J) \cdot \tilde{u}(J) = \tilde{g}(J) + \tilde{h}(J) \ , \qquad J : real, \tag{1}$$

where $\quad \tilde{K}_0(J) = \int_{-\infty}^\infty dt \ e^{-iJt} \ K_0(t) = \dfrac{2\pi}{\sqrt{1+J^2}} \ ;$

$$\tilde{h}(J) = \int_0^\infty dx \ e^{-iJx} \cdot 2\pi = \dfrac{2\pi}{iJ} \ , \qquad Im J < 0 \quad : a \ `-` \ function,$$

and
$$\tilde{g}(J) = \int_{-\infty}^0 dx \ e^{-iJx} \ g(x) \ : \quad an \ unknown \ `+` \ function.$$

In order to be able to work with $J : real$ we "move" the pole

of $\tilde{h}(J)$ to the upper half plane by $i\varepsilon$:

$$\tilde{h}(J) = \lim_{\varepsilon \to 0^+} \dfrac{2\pi}{i(J-i\varepsilon)} \ \cdot$$
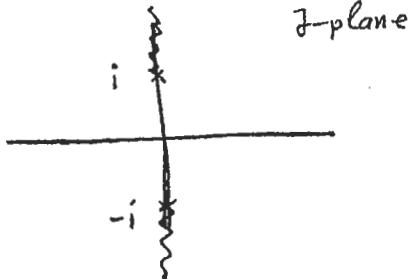
Then Eq. (1) reads

$$\underset{-}{\dfrac{2\pi}{\sqrt{1+J^2}}} \cdot \tilde{u}(J) = \underset{+}{\tilde{g}(J)} + \underset{-}{\dfrac{2\pi}{i(J-i\varepsilon)}} \tag{1a}$$

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We factorize $\dfrac{2\pi}{\sqrt{1+\jmath^2}}$ as $2\pi \dfrac{1}{\sqrt{\jmath+i}} \dfrac{1}{\sqrt{\jmath-i}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad +\qquad\quad -$

where the branch cuts for the branch points $\jmath = \pm i$ are chosen
as shown here :

$\jmath$-plane

Eq. (1a) is written as

$$\underset{-}{\frac{2\pi}{\underset{-}{\sqrt{\jmath-i}}}} \ \underset{+}{\tilde{u}(\jmath)} = \underset{+}{\sqrt{\jmath+i}} \ \tilde{g}(\jmath) + \underset{-}{\frac{2\pi}{i(\jmath-i\varepsilon)}} \ \underset{+}{\sqrt{\jmath+i}} \qquad\qquad (1b)$$

We further decomposed the "mixed" $(\ominus\oplus)$ last term to $+$ and $-$ parts:

$$\frac{2\pi}{i(\jmath-i\varepsilon)} \cdot \sqrt{\jmath+i} = \underbrace{\frac{2\pi}{i(\jmath-i\varepsilon)} \left(\sqrt{\jmath+i} - \sqrt{i\varepsilon+i} \right)}_{\substack{\text{no pole at } \jmath=i\varepsilon; \\ + \text{ function}}} + \underbrace{\frac{2\pi}{i(\jmath-i\varepsilon)} \sqrt{i\varepsilon+i}}_{\substack{\text{only a pole at } \jmath=i\varepsilon; \\ - \text{ function}}}$$

Equation (1b) is recast to

$$\underset{-}{\frac{2\pi}{\underset{-}{\sqrt{\jmath-i}}}} \underset{-}{\tilde{u}(\jmath)} - \underset{-}{\frac{2\pi}{i(\jmath-i\varepsilon)}} \sqrt{i\varepsilon+i} = \underset{+}{\sqrt{\jmath+i}} \ \underset{+}{\tilde{g}(\jmath)} + \underbrace{\frac{2\pi}{i(\jmath-i\varepsilon)} \left(\sqrt{\jmath+i} - \sqrt{i\varepsilon+i} \right)}_{+}$$

$$\equiv E(\jmath) : \text{entire function}$$

From the left-hand side of this equation, it follows that

$E(\jmath) \to 0$ as $\jmath \to \infty$ for $\text{Im}\,\jmath < 0$; hence $E(\jmath) \equiv 0$ in the lower half plane.

From the right-hand side, it then follows that $\underline{E(\jmath) \equiv 0}$ everywhere.

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Thus,

$$0 = \frac{2\pi}{\sqrt{\jmath-i}}\ \tilde{u}(\jmath) - \frac{2\pi}{i(\jmath-i\varepsilon)}\ \sqrt{i(1+\varepsilon)} = \sqrt{\jmath+i}\ \tilde{g}(\jmath) + \frac{2\pi}{i(\jmath-i\varepsilon)}\left[\sqrt{\jmath+i} - \sqrt{i(1+\varepsilon)}\right]$$
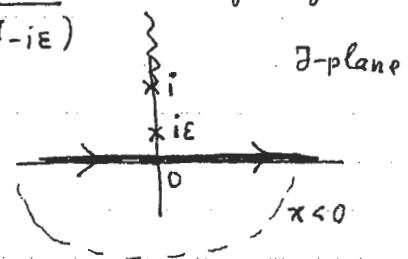
Solve for $\tilde{u}(\jmath)$:

$$\tilde{u}(\jmath) = \sqrt{i(1+\varepsilon)}\ \frac{\sqrt{\jmath-i}}{i(\jmath-i\varepsilon)} .$$

Invert to get $u(x)$:

$$u(x) = \int_{-\infty}^{\infty} \frac{d\jmath}{2\pi}\ e^{i\jmath x}\ \tilde{u}(\jmath) = \sqrt{i(1+\varepsilon)} \int_{-\infty}^{\infty} \frac{d\jmath}{2\pi}\ e^{i\jmath x}\ \frac{\sqrt{\jmath-i}}{i(\jmath-i\varepsilon)}$$
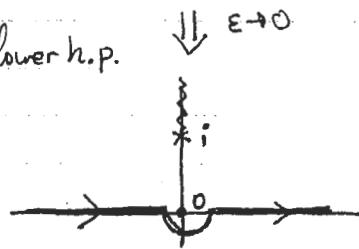
(the integrand has no singularity in lower h.p.)

$\jmath$-plane

where $\sqrt{i(1+\varepsilon)} = \sqrt{\jmath+i}\Big|_{\jmath=i\varepsilon} = e^{i\pi/2 \cdot 1/2}\cdot |1+\varepsilon|^{1/2}\Big|_{\varepsilon\to 0} = e^{i\pi/4}$.

<u>Clearly</u> $u(x) \equiv 0$ for $x<0$ by closing the path in the lower h.p. $\Downarrow \varepsilon\to 0$

$$u(x) = e^{i\pi/4} \int_{-\infty}^{\infty} \frac{d\jmath}{2\pi}\ e^{i\jmath x}\ \frac{\sqrt{\jmath-i}}{i\jmath} , \qquad x>0 ;$$

the path is <u>below</u> the pole at $\jmath=0$. $\rightarrow$

Let $\jmath' = \jmath-i$:
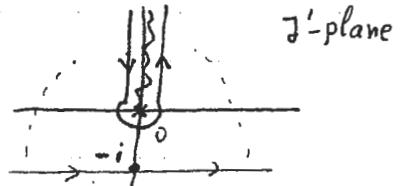
"Trick" $\searrow$

$$u(x) = \frac{e^{i\pi/4}}{i} \int_{-\infty-i}^{\infty-i} \frac{d\jmath'}{2\pi}\ e^{-x}\ e^{i\jmath' x}\ \frac{\sqrt{\jmath'}}{\jmath'+i}$$

$$\frac{\partial u}{\partial x} = \frac{i\,e^{i\pi/4}}{i} e^{-x} \int_{-\infty-i}^{\infty-i} \frac{d\jmath'}{2\pi}\ e^{i\jmath' x}\ \sqrt{\jmath'} \quad : \text{ the integrand has no pole at } \jmath'=-i.$$

We wrap the contour around the branch cut

emanating from $\jmath'=0$.

$\jmath'$-plane

$$\frac{\partial u}{\partial x} = i\frac{e^{i\pi/4}}{i} e^{-x} \int_0^\infty \frac{d(iy)}{2\pi} e^{-yx} \sqrt{e^{i\pi/2}\cdot y} \quad \longleftarrow \quad i\frac{e^{i\pi/4}}{i} e^{-x} \int_0^\infty \frac{d(iy)}{2\pi} e^{-yx} \sqrt{e^{i(2n+\pi/2)}\cdot y}$$

(set $\mathcal{J}' = e^{i\pi/2} y$ on right side of branch cut)

(set $\mathcal{J}' = e^{i(\frac{\pi}{2}+2n)}\cdot y$ on left side of branch cut)

$$= -\frac{i}{i} e^{-x} \int_0^\infty \frac{dy}{2\pi} e^{-y\cdot x} \sqrt{y} \quad \longleftarrow \quad e^{-x} \int_0^\infty \frac{dy}{2\pi} e^{-xy} \sqrt{y}$$

$$= -2 e^{-x} \int_0^\infty \frac{dy}{2\pi} e^{-xy} \sqrt{y} = -\cancel{2} e^{-x} \cdot \frac{1}{\cancel{2}\pi} \frac{1}{x^{3/2}} \underbrace{\int_0^\infty dt\, e^{-t}\cdot t^{3/2-1}}_{\Gamma(3/2) = \frac{\sqrt{\pi}}{2}}$$

$$= -\frac{1}{2\sqrt{\pi}} \frac{e^{-x}}{x^{3/2}}$$

So, $\qquad \dfrac{\partial u}{\partial x} = -\dfrac{1}{2\sqrt{\pi}} \dfrac{e^{-x}}{x^{3/2}} \quad, \qquad x > 0$

while $\qquad \cancel{\dfrac{\partial u}{\partial x}} \qquad u(x \to +\infty) = 0.$

Hence,

$$u(x) = \int_\infty^x dx'\, \frac{\partial u}{\partial x'} = \frac{1}{2\sqrt{\pi}} \int_x^\infty dx'\, \frac{e^{-x'}}{x'^{3/2}} = \frac{(-2)}{2\sqrt{\pi}} \int_x^\infty d(x'^{-1/2})\, e^{-x'}$$

$$= \frac{-1}{\sqrt{\pi}} \left[-\frac{e^{-x}}{x^{1/2}} + \int_x^\infty dx'\, \frac{e^{-x'}}{x'^{1/2}} \right] = \frac{-1}{\sqrt{\pi}} \left[-\frac{e^{-x}}{\sqrt{x}} + 2 \int_{x'=x}^\infty d\sqrt{x'}\, e^{-(\sqrt{x'})^2} \right]$$

$$= \frac{e^{-x}}{\sqrt{\pi x}} - \underbrace{\frac{2}{\sqrt{\pi}} \int_{\sqrt{x}}^\infty dz\, e^{-z^2}}_{\equiv \text{ erfc}(\sqrt{x})\,:\, \text{``error function,''}}$$

$\equiv$ erfc($\sqrt{x}$): "error function," known from probability theory (complementary)

$$\therefore \qquad u(x) = \frac{e^{-x}}{\sqrt{\pi x}} - \text{erfc}(\sqrt{x})\,, \qquad x > 0$$

$$\int_{-\infty}^{\infty} \frac{d\partial}{2\pi} \, e^{i\partial x} \; X(\partial) \, F(\partial) = f(x), \qquad x>0,$$

$$\int_{-\infty}^{\infty} \frac{d\partial}{2\pi} \, e^{i\partial x} \; F(\partial) = g(x), \qquad x<0.$$

Define

$$H(x) \stackrel{.}{=} \begin{cases} 0, & x>0, \\[4pt] \displaystyle\int_{-\infty}^{\infty} \frac{d\partial}{2\pi} \, e^{i\partial x} \; X(\partial) \, F(\partial), & x<0, \end{cases}$$

$$L(x) = \begin{cases} 0, & x<0, \\[4pt] \displaystyle\int_{-\infty}^{\infty} \frac{d\partial}{2\pi} \, e^{i\partial x} \; F(\partial), & x>0. \end{cases}$$

The original dual equations are thus defined for **all** real x:

$$\int_{-\infty}^{\infty} \frac{d\partial}{2\pi} \, e^{i\partial x} \; X(\partial) \, F(\partial) = f(x) + H(x), \qquad -\infty < x < +\infty,$$

$$\int_{-\infty}^{\infty} \frac{d\partial}{2\pi} \, e^{i\partial x} \; F(\partial) = g(x) + L(x), \qquad -\infty < x < +\infty.$$

By taking the FT of both sides of these equations we get

$$\left. \begin{array}{l} X(\partial)\, F(\partial) = \tilde{f}(\partial) + \tilde{H}(\partial) \\[4pt] F(\partial) = \tilde{g}(\partial) + \tilde{L}(\partial) \end{array} \right\} \;\Rightarrow\; X(\partial)\,\big[\, \underset{+}{\tilde{g}(\partial)} + \underset{-}{\overset{\text{unknown}}{\tilde{L}(\partial)}} \,\big] = \underset{-}{\tilde{f}(\partial)} + \underset{+}{\overset{\text{unknown}}{\tilde{H}(\partial)}},$$

and ∂ is assumed to lie on the real axis or on a line parallel to the real

axis. Let us definitely assume that ∂ lies on the real axis.

We invert the last relation to get

$$\int_{-\infty}^{\infty} \frac{d\partial}{2\pi} \, e^{i\partial x} \, \chi(\partial) \, [\, \underset{+}{\tilde{g}(\partial)} + \underset{-}{\tilde{L}(\partial)} \,] = \int_{-\infty}^{\infty} \frac{d\partial}{2\pi} \, e^{i\partial x} \, [\, \underset{-}{\tilde{f}(\partial)} + \underset{+}{\tilde{H}(\partial)} \,] ,$$

or

$$\int_{-\infty}^{\infty} dx' \, K(x-x') \, g(x') + \int_{0}^{\infty} dx' \, K(x-x') \cdot L(x') = f(x) + H(x).$$

For $\underline{x > 0}$, $H(x) = 0$ and

$$\int_{0}^{\infty} dx' \, K(x-x') \cdot L(x') = f(x) - \int_{-\infty}^{0} dx' \, K(x-x') \, g(x'),$$

or,

$$\int_{0}^{\infty} dx' \, K(x-x') \cdot L(x') = f(x) + g_1(x) \quad , \quad g_1(x) \equiv -\int_{-\infty}^{0} dx' \, K(x-x') \, g(x'),$$

where $L(x)$ is unknown, and $f(x)$ and $g_1(x)$ are known. Hence, the

two original integral equations reduce to a $\underline{single}$ integral equation which

is a Wiener-Hopf equation of the first kind. This last equation can be

solved by the method of factorization. For this purpose, we return to the original

relation among FTs,

$$\chi(\partial) \, [\, \underset{+}{\tilde{g}(\partial)} + \underset{-}{\tilde{L}(\partial)} \,] = \underset{-}{\tilde{f}(\partial)} + \overset{\overbrace{unknown}}{\underset{+}{\tilde{H}(\partial)}} \quad , \quad \partial : real.$$

(with "unknown" labels pointing to $\tilde{L}(\partial)$ and $\tilde{H}(\partial)$)

By setting

$$\chi(\partial) = \frac{P_-(\partial)}{P_+(\partial)} ,$$

we get

$$P_-(\partial) \, \underset{-}{\tilde{L}(\partial)} = P_+(\partial) \, \underset{-}{\tilde{f}(\partial)} - P_-(\partial) \, \underset{+}{\tilde{g}(\partial)} + P_+(\partial) \, \underset{+}{\tilde{H}(\partial)} .$$

Since $\tilde{f}(\mathfrak{z})$ and $\tilde{g}(\mathfrak{z})$ are known, we decompose the RHS as

$$\underset{-}{P_+(\mathfrak{z})\ \tilde{f}(\mathfrak{z})} - \underset{+}{P_-(\mathfrak{z})\cdot\tilde{g}(\mathfrak{z})} \equiv R_+(\mathfrak{z}) + R_-(\mathfrak{z}),$$

which in turn entails

$$\underset{-}{P_-(\mathfrak{z})\ \tilde{L}(\mathfrak{z}) - R_-(\mathfrak{z})} = \underset{+}{R_+(\mathfrak{z}) + P_+(\mathfrak{z})\ \tilde{H}(\mathfrak{z})} \equiv E(\mathfrak{z}), \quad \mathfrak{z}:\text{real}.$$

$$\underbrace{\qquad}_{\substack{\text{entire fcn}\\ (\text{for cmpx } \mathfrak{z})}}$$

The function $E(\mathfrak{z})$ is determined via taking the limits of

$P_-\tilde{L} - R_-$ and $R_+ + P_+\tilde{H}$ as $|\mathfrak{z}|\to\infty$ in the respective half planes, using

the fact that $\tilde{H}, \tilde{L} \to 0$ as $|\mathfrak{z}|\to\infty$. Finally,

$$\tilde{L}(\mathfrak{z}) = \frac{E(\mathfrak{z}) + R_-(\mathfrak{z})}{P_-(\mathfrak{z})}, \qquad \tilde{H}(\mathfrak{z}) = \frac{E(\mathfrak{z}) - R_+(\mathfrak{z})}{P_+(\mathfrak{z})}.$$

Note : The procedure given here merely shows that the dual integral equations

reduce to a __single__ integral equation of the first kind. Hence, the $\overset{\text{solvability}}{\text{issues}}$ that

"plague" equations of the first kind are still inherent in the dual integral

equations. One case where the dual equations are explicitly solved is

studied in connection with the mixed boundary-value problem of

Laplace's equation. (compare with classnotes and __verify__ this statement!)