Solutions to Homework 5

For Prob. 15, I assumed some familiarity with the properties of Fourier transforms in the complex (k-) plane.

(15) $$u(x) = f(x) + \lambda \int_{-\infty}^{\infty} dy \; e^{-|x-y|} \; u(y), \qquad -\infty < x < \infty.$$

(a) $$f(x) = \begin{cases} x, & x > 0 \\ 0, & x < 0 \end{cases}.$$

The Fourier transform of $f(x)$ is

$$\tilde{f}(k) = \int_0^{+\infty} dx \; e^{-ikx} \; x = \frac{d}{d(-ik)} \int_0^{+\infty} dx \; e^{-ikx} = \frac{d}{d(-ik)} \frac{1}{ik} = -\frac{1}{k^2},$$

assuming $k = \xi - i\eta$, $\eta > 0$, i.e., restricting k in the lower half of the k complex plane; $\mathrm{Im}\, k < 0$.

With this in mind, we take the F.T. of both sides of the given equation:

$$\tilde{u}(k) = \tilde{f}(k) + \lambda \tilde{K}(k) \tilde{u}(k),$$

where $$\tilde{K}(k) = \int_{-\infty}^{\infty} dx \; e^{-|x|} \; e^{-ikx} = \frac{2}{1+k^2}.$$

Hence,

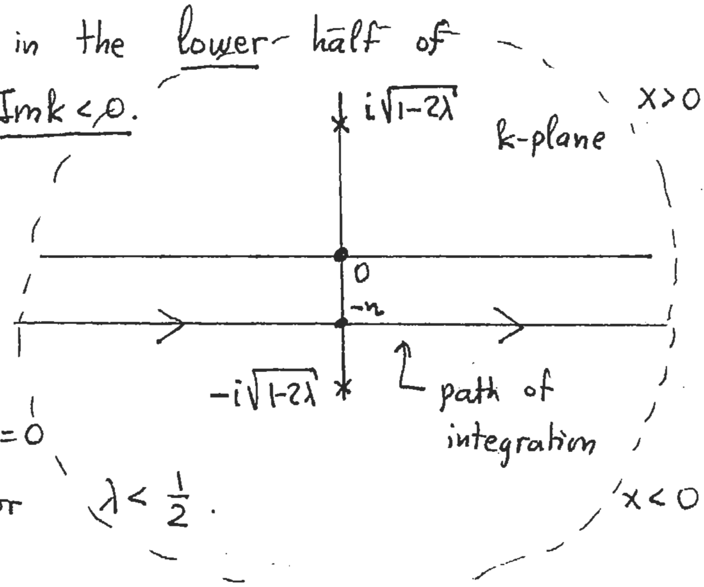
$$\tilde{u}(k) = -\frac{1}{k^2} + \frac{2\lambda}{1+k^2}\,\tilde{u}(k) \quad , \qquad \operatorname{Im} k < 0 ,$$

$$\rightarrow \quad \tilde{u}(k) = \frac{-1/k^2}{1-\frac{2\lambda}{1+k^2}} = -\frac{1}{k^2}\,\frac{1+k^2}{1+k^2-2\lambda} .$$

A particular solution to the given equation is thus obtained as

$$u_p(x) = \int_{-\infty-i\eta}^{+\infty-i\eta} \frac{dk}{2\pi}\, e^{ikx}\, \tilde{u}(k) = -\int_{-\infty-i\eta}^{+\infty-i\eta} \frac{dk}{2\pi}\, e^{ikx}\, \frac{1}{k^2}\, \frac{1+k^2}{1+k^2-2\lambda} \quad , \qquad \eta > 0.$$

Notice that the path of integration lies in the lower half of the k plane, because of the restriction $\operatorname{Im} k < 0$.



The integrand has a double pole at $k=0$ and simple poles at $k = \pm i\sqrt{1-2\lambda}$ for $\lambda < \frac{1}{2}$.

It follows that, for $\lambda < \frac{1}{2}$ and the path being sufficiently close to the real axis (so that it lies between the pole at $-i\sqrt{1-2\lambda}$ and the real axis),

$$u_p(x) = -2\pi i\,\frac{1}{2\pi}\left\{ \operatorname*{Res}_{k=0}\left[\frac{e^{ikx}}{k^2}\,\frac{1+k^2}{1+k^2-2\lambda}\right] + \operatorname*{Res}_{k=i\sqrt{1-2\lambda}}\left[\frac{e^{ikx}}{k^2}\,\frac{1+k^2}{1+k^2-2\lambda}\right]\right\}, \quad x>0,$$

$$u_p(x) = 2\pi i\cdot\frac{1}{2\pi}\operatorname*{Res}_{k=-i\sqrt{1-2\lambda}}\left[\frac{e^{ikx}}{k^2}\,\frac{1+k^2}{1+k^2-2\lambda}\right] \quad , \qquad x<0,$$

by closing the path in the upper half plane for $x>0$ and in the lower half plane for $x<0$.

We need to calculate the residue at $k=0$:

$$\frac{e^{ikx}}{k^2}\ \frac{1+k^2}{1+k^2-2\lambda} \underset{k\to 0}{\sim} \frac{1+ikx}{k^2}\cdot \underbrace{F(k^2)}_{\text{fcn of } k^2 \text{ only, analytic at } k^2=0.} , \quad F(k^2) \equiv \frac{1+k^2}{1+k^2-2\lambda} ;$$

For our purposes, it suffices to take

$$\frac{e^{ikx}}{k^2}\ \frac{1+k^2}{1+k^2-2\lambda} \underset{k\to 0}{\sim} \frac{1+ikx}{k^2}\cdot F(0) = \frac{1}{1-2\lambda}\left(\frac{1}{k^2}+\frac{ix}{k}\right)$$

$$\to \operatorname*{Res}_{k=0}\left[\frac{e^{ikx}}{k^2}\ \frac{1+k^2}{1+k^2-2\lambda}\right] = \frac{ix}{1-2\lambda} \qquad (\lambda \neq \tfrac{1}{2}).$$

Hence, for $x>0$,

$$u_p(x) = -i\left[\frac{ix}{1-2\lambda} + \frac{e^{-\sqrt{1-2\lambda}\,x}}{(i\sqrt{1-2\lambda})^2}\ \frac{1-1+2\lambda}{2i\sqrt{1-2\lambda}}\right]$$

$$= \frac{x}{1-2\lambda} + \frac{\lambda}{(1-2\lambda)^{3/2}}\,e^{-\sqrt{1-2\lambda}\,x}, \qquad x>0,$$

$$u_p(x) = i\cdot\frac{e^{\sqrt{1-2\lambda}\,x}}{+(1-2\lambda)}\ \frac{\cancel{2}\lambda}{+\cancel{2}i\sqrt{1-2\lambda}} = \frac{\lambda e^{\sqrt{1-2\lambda}\,x}}{(1-2\lambda)^{3/2}}, \qquad x<0.$$

The general solution is

$$u(x) = u_h(x) + u_p(x), \qquad \underline{0<\lambda<\tfrac{1}{2}},$$

where

$$u_h(x) = C_1\,e^{-\sqrt{1-2\lambda}\,x} + C_2\,e^{+\sqrt{1-2\lambda}\,x}, \qquad C_1, C_2 : \text{arbitrary consts.},$$

is the solution to the homogeneous equation.

For $\underline{\lambda \le 0}$, $\quad u(x) = u_p(x)$,

since $\quad C_1 \equiv 0 \equiv C_2 \quad$ because λ does <u>not</u> belong to the kernel spectrum $\{\lambda > 0\}$.

For $\underline{\lambda \ge \frac{1}{2}}$, $\quad u(x) = u_h(x) + u_p(x)$, where one continues analytically the square-root functions as

$$\sqrt{1-2\lambda} \rightarrow -i\sqrt{2\lambda-1} \quad \text{etc.}$$

(b) $\quad f(x) = x, \qquad -\infty < x < +\infty$

$$f(x) = \underbrace{\begin{cases} x, & x>0 \\ 0, & x<0 \end{cases}}_{f_1(x)} + \underbrace{\begin{cases} 0, & x>0 \\ x, & x<0 \end{cases}}_{f_2(x)}.$$

Define $\quad u_i(x) \quad (i=1,2) \quad$ so that each one satisfies

$$u_i(x) = f_i(x) + \lambda \int_{-\infty}^{\infty} dy \; e^{-|x-y|} \, u_i(y).$$

The solution to the given integral equation then reads as

$$u(x) = u_1(x) + u_2(x).$$

From part (a) above, a particular solution $\quad u_{1,p}(x) \quad$ is

$$u_{1,p}(x) = \begin{cases} \dfrac{x}{1-2\lambda} + \dfrac{\lambda}{(1-2\lambda)^{3/2}} e^{-\sqrt{1-2\lambda}\, x}, & x>0, \\[2ex] \dfrac{\lambda}{(1-2\lambda)^{3/2}} e^{\sqrt{1-2\lambda}\, x}, & x<0, \end{cases} \qquad \text{when} \quad \lambda < \tfrac{1}{2}.$$

In order to find a particular solution $u_{2,p}(x)$, we need to calculate the Fourier transform

$$\tilde{f}_2(k) = \int_{-\infty}^{+\infty} dx \; f_2(x) \, e^{-ikx} = \int_{-\infty}^{0} dx \cdot x \; e^{-ikx} = \int_{0}^{+\infty} dx \, (-x) \, e^{-i(-k)x}$$

$$= -\int_{0}^{+\infty} dx \; x \; e^{-i(-k)x} = -\tilde{f}_1(-k) = +\frac{1}{(-k)^2} = \frac{1}{k^2}, \qquad \mathrm{Im}(-k) < 0$$

$$\Longrightarrow \quad \tilde{f}_2(k) = \frac{1}{k^2}, \qquad \mathrm{Im}\,k > 0,$$

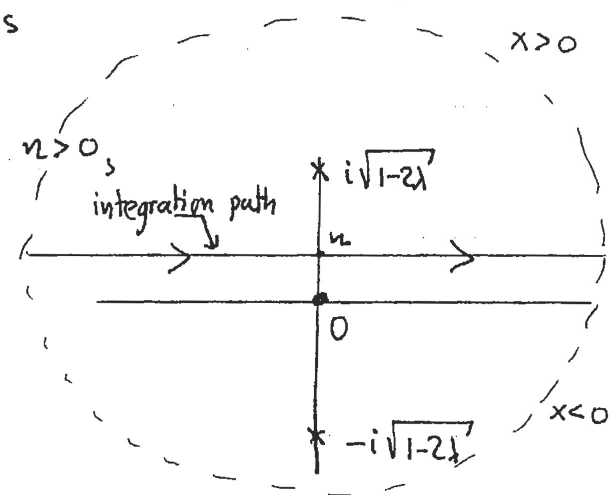
i.e., by restricting k in the <u>upper</u> half plane. It follows that

$$\tilde{u}_2(k) = \frac{\tilde{f}_2(k)}{1 - \frac{2\lambda}{1+k^2}} = \frac{1}{k^2} \frac{1+k^2}{1+k^2-2\lambda}.$$

$(\lambda < \frac{1}{2})$

A particular solution $u_{2,p}(x)$ is thus obtained as

$$u_{2,p}(x) = \int_{-\infty+i\eta}^{+\infty+i\eta} \frac{dk}{2\pi} \frac{1}{k^2} \frac{1+k^2}{1+k^2-2\lambda} e^{ikx},$$



$$= \begin{cases} 2\pi i \cdot \frac{1}{2\pi} \cdot \operatorname*{Res}_{k=i\sqrt{1-2\lambda}} \left[\frac{1}{k^2} \frac{1+k^2}{1+k^2-2\lambda} e^{ikx} \right], & x>0 \\[2ex] -2\pi i \frac{1}{2\pi} \cdot \left\{ \operatorname*{Res}_{k=0} \left[\frac{1}{k^2} \frac{1+k^2}{1+k^2-2\lambda} e^{ikx} \right] + \operatorname*{Res}_{k=-i\sqrt{1-2\lambda}} \left[\frac{1}{k^2} \frac{1+k^2}{1+k^2-2\lambda} e^{ikx} \right] \right., & x<0. \end{cases}$$

It follows that

$$u_{2,p}(x) = \begin{cases} -\frac{\lambda}{(1-2\lambda)^{3/2}} e^{-\sqrt{1-2\lambda}\,x}, & x>0 \\[2ex] \frac{x}{1-2\lambda} - \frac{\lambda}{(1-2\lambda)^{3/2}} e^{\sqrt{1-2\lambda}\,x}, & x<0. \end{cases} \qquad (\lambda < \tfrac{1}{2}).$$

The general solution to the given equation is as follows

$$
\begin{cases}
u(x) = u_h(x) + u_{1,p}(x) + u_{2,p}(x) \quad , & 0 < \lambda < \frac{1}{2} \,, \quad (\mathrm{I}) \\
u(x) = u_{1,p}(x) + u_{2,p}(x) \quad , & \lambda \leq 0 \,, \quad (\mathrm{II}) \\
u(x) = u_h(x) + u_{1,p}(x) + u_{2,p}(x) \quad , & \lambda \geq \frac{1}{2} \,, \quad (\mathrm{III})
\end{cases}
$$

where

$$u_h(x) = C_1 \, e^{-\sqrt{1-2\lambda}\, x} + C_2 \, e^{\sqrt{1-2\lambda}\, x} \quad \text{in} \quad (\mathrm{I}),$$

or,

$$u_h(x) = C_1 \, e^{i\sqrt{2\lambda-1}\, x} + C_2 \, e^{-i\sqrt{2\lambda-1}\, x} \quad \text{in} \quad (\mathrm{III}).$$

(16) Suppose that $\{\lambda_n\}$ is the sequence of the eigenvalues of $K(x,y)$ (ordered by increasing magnitude), with corresponding eigenfunctions $\{u_n\}$. Then,

$$K(x,y) = \sum_n \frac{u_n(x)\, u_n(y)}{\lambda_n} \,, \qquad a \leq x,y \leq b, \quad \underline{\text{with}} \quad \int_a^b dx \; u_n(x)\, u_\ell(x) = \delta_{n\ell}.$$

Let us assume, without loss of generality, that $\check{\lambda} = \lambda_m$ (for some m) and that $\check{\lambda}$ has multiplicity 1 (there is only <u>one</u> corresponding $u_n = u_m$).

Then,

$$\check{\lambda} \int_a^b dy \; K(x,y)\, u(y) = \lambda_m \int_a^b dy \sum_n \frac{u_n(x)\, u_n(y)}{\lambda_n} \, u(y)$$

$$= \sum_n \frac{\lambda_m}{\lambda_n} u_n(x) \int_a^b dy \; u_n(y)\, u(y) = \sum_n c_n \left(\frac{\lambda_m}{\lambda_n} \right) u_n(x),$$

where

$$c_n \equiv \int_a^b dy \; u_n(y)\, u(y).$$

The given integral equation becomes

$$u(x) = f(x) + \sum_n c_n \left(\frac{\lambda_m}{\lambda_n}\right) u_n(x), \qquad a \le x \le b.$$

Multiply both sides by $u_p(x)$ and integrate $\int_a^b dx\,(\cdots)$:

$$\underbrace{\int_a^b dx\, u_p(x)\, u(x)}_{c_p} = \int_a^b dx\, u_p(x)\, f(x) + \sum_n c_n \left(\frac{\lambda_m}{\lambda_n}\right) \underbrace{\int_a^b dx\, u_p(x)\, u_n(x)}_{\delta_{pn}}$$

$$\Rightarrow \quad c_p = \int_a^b dx\, u_p(x)\, f(x) + c_p \frac{\lambda_m}{\lambda_p}$$

$$\Rightarrow \quad c_p\left(1 - \frac{\lambda_m}{\lambda_p}\right) = \int_a^b dx\, u_p(x)\, f(x).$$

So, if there is a $u(x)$ that satisfies the original equation, then this last equation has to hold. In particular, for $p = m$ one gets

$$0 = \int_a^b dx\, u_m(x)\, f(x).,$$

i.e. $f(x)$ is orthogonal to the eigenfunction $u_m(x)$ corresponding to $\check{\lambda} = \lambda_m$.

If the multiplicity of $\check{\lambda}$ is k, $k \geq 2$, then the above argument can be repeated for each eigenfunction $u_{m,j}(x)$, $j = 1, 2, \ldots, k$.

(5)
$$u(x) = f(x) + \lambda \int_0^1 dy \; \min\{x,y\} \cdot u(y)$$
$$= f(x) + \lambda \left[\int_0^x dy \; y u(y) + \int_x^1 dy \; x u(y) \right]$$

(a) $f \equiv 0$:
$$u(x) = \lambda \left[\int_0^x dy \; y u(y) + \int_x^1 dy \; x u(y) \right]$$

From this equation, we get $\vee$ $\underline{u(0) = 0}$.

$$u'(x) = \lambda x u(x) - \lambda x u(x) + \lambda \int_x^1 dy \; u(y) = \lambda \int_x^1 dy \; u(y) \quad , \quad \underline{u'(1) = 0}$$

$\Rightarrow$ $u''(x) = -\lambda u(x)$.

Hence,
$$\begin{cases} u''(x) + \lambda u(x) = 0 \\ u(0) = 0 = u'(1) \end{cases}$$

$u(x) = A \cos(\sqrt{\lambda}\, x) + B \sin(\sqrt{\lambda}\, x)$

$u(0) = 0 \rightarrow A = 0$

$u'(1) = 0 \rightarrow \cos(\sqrt{\lambda}) = 0 \rightarrow \sqrt{\lambda} = (2n+1)\frac{\pi}{2} \rightarrow \lambda = \lambda_n = (2n+1)^2 \frac{\pi^2}{4}$, $n = 0, 1, 2, \ldots$

Eigenfunctions: $u(x) = u_n(x) = B_n \sin\left[(2n+1) \frac{\pi x}{2} \right]$.

(b) The lowest eigenvalue is $\lambda = \lambda_0 = \pi^2/4$.

The iteration series will converge for $|\lambda| < \pi^2/4$.

(c)
$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n^2} = \int_0^1 dx \; \underbrace{K_2(x,x)}_{\substack{\text{iterate} \\ \text{of } K(x,y)}} = \int_0^1 dx \int_0^1 dy \; K(x,y)\, K(y,x) = \int_0^1 dx \int_0^1 dy \; K(x,y)^2$$
$$= 2 \int_0^1 dx \int_0^x dy \; y^2 = 2 \int_0^1 dx \; \frac{x^3}{3} = \frac{2}{3} \, \frac{1}{4} = \frac{1}{6} .$$