18.307: Integral Equations M.I.T. Department of Mathematics

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Solutions to Problem Set 4

(11) $$D(\lambda) = \sum_{n=0}^{\infty} \frac{D^{(n)}(0)}{n!}\lambda^n, \qquad D^{(n)}(0) = (-1)^n \int_a^b dx_1 \int_a^b dx_2 \cdots \int_a^b dx_n\, K\begin{pmatrix} x_1 & x_2 \cdots x_n \\ x_1 & x_2 \cdots x_n \end{pmatrix},$$

where $D(\lambda)$ is the determinant corresponding to the Fredholm equation

$$u(x) = f(x) + \lambda \int_a^b dy\, K(x,y)\, u(y),$$

$K(x,y)$ is the kernel, and

$$K\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix} = \begin{vmatrix} K(x_1,y_1) & K(x_1,y_2) & \cdots & K(x_1,y_n) \\ K(x_2,y_1) & K(x_2,y_2) & \cdots & K(x_2,y_n) \\ \cdots & \cdots & & \cdots \\ K(x_n,y_1) & K(x_n,y_2) & & K(x_n,y_n) \end{vmatrix}, \qquad D^{(0)}(0) = 1.$$

(a) $K(x,y) = \pm 1, \qquad a=0,\ b=1.$

$$D^{(1)}(0) = -\int_0^1 dx\, K(x,x) = \mp 1,$$

$\underline{n\geq 2}$: $$D^{(n)}(0) = (-1)^n \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \overbrace{\begin{vmatrix} K(x_1,x_1) & K(x_1,x_2) & \cdots & K(x_1,x_n) \\ K(x_2,x_1) & K(x_2,x_2) & \cdots & K(x_2,x_n) \\ \cdots & \cdots & & \cdots \\ K(x_n,x_1) & K(x_n,x_2) & \cdots & K(x_n,x_n) \end{vmatrix}}^{K\begin{pmatrix} x_1 & x_2 \cdots x_n \\ x_1 & x_2 \cdots x_n \end{pmatrix}},$$

where $$K\begin{pmatrix} x_1 & x_2 \cdots x_n \\ x_1 & x_2 \cdots x_n \end{pmatrix} = \begin{vmatrix} \pm 1 & \pm 1 & \cdots & \pm 1 \\ \pm 1 & \pm 1 & \cdots & \pm 1 \\ \cdots & \cdots & & \cdots \\ \pm 1 & \pm 1 & \cdots & \pm 1 \end{vmatrix} = 0,$$ because the column vectors are linearly dependent. Hence,

$$D(\lambda) = 1 \mp \lambda.$$

(b) $K(x,y) = g(x)\, g(y), \qquad x \in [a,b]$

$$D^{(1)}(0) = -\int_a^b dx \quad K(x,x) = -\int_a^b dx \quad g(x)^2,$$

$$D^{(n\geq 2)}(0) = (-1)^n \int_a^b dx_1 \int_a^b dx_2 \cdots \int_a^b dx_n \quad K\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}, \qquad \text{where}$$

$$K\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} = \begin{vmatrix} g(x_1)\,g(x_1) & g(x_1)\,g(x_2) & \cdots & g(x_1)\,g(x_n) \\ g(x_2)\,g(x_1) & g(x_2)\,g(x_2) & \cdots & g(x_2)\,g(x_n) \\ \cdots & \cdots & & \cdots \\ g(x_n)\,g(x_1) & g(x_n)\,g(x_2) & \cdots & g(x_n)\,g(x_n) \end{vmatrix}.$$

The mth column vector is equal to
$$g(x_m) \begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_n) \end{bmatrix}.$$

It follows that <u>any</u> <u>two</u> column vectors are linearly dependent, and hence

$$K\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} = 0, \qquad n \geq 2.$$

$$\rightarrow \quad D(\lambda) = 1 - \lambda \int_a^b dx \quad g(x)^2$$

(c) $K(x,y) = x + y, \qquad a=0,\ b=1.$

$$D^{(1)}(0) = -\int_0^1 dx \quad 2x = -1.$$

<u>$n \geq 2$:</u>

$$K\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} = \begin{vmatrix} 2x_1 & x_1+x_2 & \cdots & x_1+x_n \\ x_2+x_1 & 2x_2 & \cdots & x_2+x_n \\ \cdots & \cdots & & \cdots \\ x_n+x_1 & x_n+x_2 & \cdots & 2x_n \end{vmatrix} =$$

$$= \begin{vmatrix} x_1-x_2 & x_2-x_3 & \cdots & x_1+x_n \\ x_1-x_2 & x_2-x_3 & \cdots & x_2+x_n \\ \vdots & \vdots & & \vdots \\ x_1-x_2 & x_2-x_3 & \cdots & 2x_n \end{vmatrix} = (x_1-x_2)(x_2-x_3)\cdots(x_{n-1}-x_n)$$

$$\cdot \begin{vmatrix} 1 & 1 & \cdots & x_1+x_n \\ 1 & 1 & \cdots & x_2+x_n \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 2x_n \end{vmatrix} = \begin{cases} (x_1-x_2)\begin{vmatrix} 1 & x_1+x_2 \\ 1 & 2x_2 \end{vmatrix} = -(x_2-x_1)^2, & n=2 \\ 0 & , n \geq 3. \end{cases}$$

It follows that

$$D^{(2)}(0) = \int_0^1 dx_1 \int_0^1 dx_2 \, [-(x_2-x_1)^2] = 1 - 7/6 = -\frac{1}{6},$$

$$D^{(n\geq 3)}(0) = 0.$$

$$D(\lambda) = 1 - \lambda - \frac{1}{2!}\frac{1}{6}\lambda^2 = 1 - \lambda - \frac{1}{12}\lambda^2.$$

(d) $K(x,y) = x^2+y^2, \quad a=0, \; b=1.$

$$D^{(1)}(0) = -\int_0^1 dx \; 2x^2 = -2\,\frac{1}{3},$$

$\underline{n \geq 2}$:

$$K\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} = \begin{vmatrix} x_1^2+x_1^2 & x_1^2+x_2^2 & \cdots & x_1^2+x_n^2 \\ x_2^2+x_1^2 & x_2^2+x_2^2 & \cdots & x_2^2+x_n^2 \\ \cdots & \cdots & & \cdots \\ x_n^2+x_1^2 & x_n^2+x_2^2 & \cdots & x_n^2+x_n^2 \end{vmatrix}$$

$$= \begin{vmatrix} (x_1^2-x_2^2) & (x_2^2-x_3^2) & \cdots & x_1^2+x_n^2 \\ x_1^2-x_2^2 & x_2^2-x_3^2 & \cdots & x_2^2+x_n^2 \\ \cdots & \cdots & & \cdots \\ x_1^2-x_2^2 & (x_2^2-x_3^2) & \cdots & x_n^2+x_n^2 \end{vmatrix} = (x_1^2-x_2^2)\cdot(x_2^2-x_3^2)\cdots(x_{n-1}^2-x_n^2)$$

$$\cdot \begin{vmatrix} 1 & 1 & \cdots & x_1^2+x_n^2 \\ 1 & 1 & \cdots & x_2^2+x_n^2 \\ \cdots & \cdots & & \\ 1 & 1 & \cdots & x_n^2+x_n^2 \end{vmatrix}$$

$$= \begin{cases} (x_1^2-x_2^2) \begin{vmatrix} 1 & x_1^2+x_2^2 \\ 1 & x_2^2+x_2^2 \end{vmatrix} = (x_1^2-x_2^2)(x_2^2-x_1^2) = -(x_2^2-x_1^2)^2, & n=2 \\ 0, & n\geq 3 \end{cases}$$

$$D^{(2)}(0) = -\int_0^1 dx_1 \int_0^1 dx_2 \, (x_2^2-x_1^2)^2 = -\left(\frac{1}{5}+\frac{1}{5}-2\cdot\frac{1}{3}\cdot\frac{1}{3}\right) = -\left(\frac{2}{5}-\frac{2}{9}\right) = -2\,\frac{4}{45} = -\frac{8}{45},$$

$D^{(n\geq 3)}(0) = 0.$ Hence,

$$D(\lambda) = 1 - \frac{2}{3}\lambda - \frac{4}{45}\lambda^2.$$

(e) $K(x,y) = xy(x+y), \quad x \in [0,1].$

$$D^{(1)}(0) = -\int_0^1 dx \; x^2 \cdot 2x = -2\cdot\frac{1}{4} = -\frac{1}{2},$$

$$K\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} = \begin{vmatrix} 2x_1^3 & x_1x_2(x_1+x_2) & x_1x_3(x_1+x_3) & \cdots & x_1x_n(x_1+x_n) \\ x_2x_1(x_2+x_1) & 2x_2^3 & x_2x_3(x_2+x_3) & \cdots & x_2x_n(x_2+x_n) \\ \cdots & & & & \\ x_nx_1(x_n+x_1) & x_nx_2(x_n+x_2) & x_nx_3(x_n+x_3) & \cdots & 2x_n^3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1(x_1-x_2)(x_1+x_2+x_1) & x_1(x_2-x_3)(x_2+x_3+x_1) & \cdots & x_1x_n(x_1+x_n) \\ x_2(x_1-x_2)(x_1+x_2+x_2) & x_2(x_2-x_3)(x_2+x_3+x_2) & \cdots & x_2x_n(x_2+x_n) \\ \cdots & \cdots & & \\ x_n(x_1-x_2)(x_1+x_2+x_n) & x_n(x_2-x_3)(x_2+x_3+x_n) & \cdots & 2x_n^3 \end{vmatrix}$$

$$= x_1x_2\cdots x_{n-1}x_n^2\,(x_1-x_2)(x_2-x_3)\cdots(x_{n-1}-x_n) \begin{vmatrix} x_1+x_2+x_1 & x_2+x_3+x_1 & \cdots & x_1+x_n \\ x_1+x_2+x_2 & x_2+x_3+x_2 & \cdots & x_2+x_n \\ \cdots & \cdots & & \cdots \\ x_1+x_2+x_n & x_2+x_3+x_n & & 2x_n \end{vmatrix}$$

$$= x_1 x_2 \cdots x_{n-1} x_n^2 \, (x_1-x_2)\,(x_2-x_3)\cdots(x_{n-1}-x_n) \begin{vmatrix} x_1-x_3 & x_2-x_4 & \cdots & x_{n-1} & x_1+x_n \\ x_1-x_3 & x_2-x_4 & \cdots & x_{n-1} & x_2+x_n \\ \cdots & \cdots & \cdots & \cdots \\ x_1-x_3 & x_2-x_4 & \cdots & x_{n-1} & 2x_n \end{vmatrix}$$

$$= x_1 x_2 \cdots x_{n-1}^2 x_n^2 \, (x_1-x_2)(x_1-x_3)(x_2-x_3)(x_2-x_4)\cdots(x_{n-1}-x_n) \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & x_1+x_n \\ 1 & 1 & \cdots & 1 & x_2+x_n \\ \cdots & \cdots & & & \cdots \\ 1 & 1 & \cdots & 1 & 2x_n \end{vmatrix}$$

$$= \begin{cases} -x_1^2 x_2^2 \, (x_1-x_2)^2, & n=2 \\ 0, & n \geq 3. \end{cases}$$

$$D^{(2)}(0) = -\int_0^1 dx_1 \int_0^1 dx_2 \;\; x_1^2 x_2^2 (x_1-x_2)^2 = -\left(2\cdot\frac{1}{5}\cdot\frac{1}{3} - 2\cdot\frac{1}{4}\cdot\frac{1}{4}\right) = -2\left(\frac{1}{15}-\frac{1}{16}\right) = -\frac{1}{120},$$

$D^{(n\geq 3)}(0) = 0.$ Hence, $\quad D(\lambda) = 1 - \frac{\lambda}{2} - \frac{\lambda^2}{240}.$

(12). $$u(x) = f(x) + \lambda \int_0^{+\infty} dx' \; K(x,x') \; u(x'), \qquad 0 \leq x.$$

Without loss of generality, I took $a=0$.

(a) Let $t = \frac{x}{1+x}$, $t' = \frac{x'}{1+x'}$, $u(x') \equiv \phi(t')$, $u(x) \equiv \phi(t)$.

$\rightarrow x' = \frac{t'}{1-t'} \rightarrow dx' = \frac{dt'}{(1-t')^2}$. For $x'=0 \rightarrow t'=0$.

The function $\frac{x}{1+x}$ is monotonically increasing; for $x' \rightarrow +\infty$, $t' \rightarrow 1$.

Suppose $f(x) \equiv F(t)$. The original equation becomes

$$\phi(t) = F(t) + \lambda \int_0^1 dt' \; \frac{1}{(1-t')^2} \; K[x(t), x'(t')] \; \phi(t'), \qquad 0 \leq t < 1.$$

Let $K(x,x') = (1-t)(1-t')\,\kappa(t,t')$. Then,

$$\phi(t) = F(t) + \lambda \int_0^1 dt' \; \frac{1-t}{1-t'} \; \kappa(t,t') \; \phi(t'), \qquad 0 \leq t < 1.$$

(b) To symmetrize the given equation "as much as possible," we multiply both sides of the last equation by $\frac{1}{1-t}$:

$$\frac{\phi(t)}{1-t} = \frac{F(t)}{1-t} + \lambda \int_0^1 dt' \quad K(t,t') \left[\frac{\phi(t')}{1-t'} \right].$$

Define

$$g(t) \equiv \frac{F(t)}{1-t} = \frac{f(x)}{\frac{1}{1+x}} = (1+x)\, f(x), \qquad \chi(t) \equiv \frac{\varphi(t)}{1-t} = (1+x)\, u(x).$$

$$\chi(t) = g(t) + \lambda \int_0^1 dt' \; K(t,t') \; \chi(t').$$

$$\|K\|^2 = \int_0^1 dt \int_0^1 dt' \; |K(t,t')|^2 = \int_0^1 dt \int_0^1 dt' \; \frac{|K(x,x')|^2}{(1-t)^2 \; (1-t')^2}$$

$$= \int_0^1 \underbrace{\frac{dt}{+(1-t)^2}}_{dx} \int_0^1 \underbrace{\frac{dt'}{+(1-t')^2}}_{dx'} |K(x,x')|^2 = \int_0^{+\infty} dx \int_0^{+\infty} dx' \; |K(x,x')|^2 = \|K\|^2.$$

Note that if $K(x,x')$ is symmetric, then $K(t,t')$ is symmetric.

$$\|g\|^2 = \int_0^1 dt \; |g(t)|^2 = \int_0^1 dt \; \frac{|F(t)|^2}{(1-t)^2} = \int_0^1 \underbrace{\frac{dt}{(1-t)^2}}_{dx} |F(t)|^2 = \int_0^{+\infty} dx \; |f(x)|^2 = \|f\|^2.$$

(What happens if $a = -\infty$ and $b = +\infty$, where (a,b) is the original range of integration?).

(13) $$\psi(x) = e^{ikx} + \int_{-\infty}^{\infty} dy \quad \frac{e^{ik|x-y|}}{2ik} \quad V(y)\,\psi(y) \,, \qquad -\infty < x < \infty .$$

Multiply both sides by $\sqrt{V(x)}$, assuming $V(x) > 0$:

$$\phi(x) = e^{ikx}\sqrt{V(x)} + \lambda \int_{-\infty}^{\infty} dy \quad K(x,y) \quad \phi(y) \,,$$

where $\phi(x) \equiv \psi(x)\sqrt{V(x)}$, $\lambda \equiv \frac{1}{k}$, $K(x,y) \equiv \frac{e^{ik|x-y|}}{2i}\sqrt{V(x)V(y)}$: symmetric.

For $k \to +\infty$, $\lambda \to 0^+$, and the function $D(\lambda)$ can be approximated as

$$D(\lambda) \sim 1 + D'(0)\cdot\lambda = 1 + \frac{1}{k}\, D'(0) \,,$$

where $$D'(0) = -\int_{-\infty}^{\infty} dx \quad K(x,x) = -\int_{-\infty}^{\infty} dx \quad \frac{|V(x)|}{2i} = -\frac{1}{2i}\int_{-\infty}^{\infty} dx \cdot |V(x)| \,.$$

We further assume that $\int_{-\infty}^{\infty} dx \; |V(x)| < \infty$.

The function $N(x,y;\lambda)$ is approximated as

$$N(x,y;\lambda) \sim K(x,y) - \frac{\lambda}{1!}\, N_1(x,y) = K(x,y) - \lambda \int_{-\infty}^{\infty} dx_1 \quad K\begin{pmatrix} x & x_1 \\ y & x_1 \end{pmatrix}$$

$$= K(x,y) - \frac{1}{k} \int_{-\infty}^{\infty} dx_1 \quad \begin{vmatrix} K(x,y) & K(x,x_1) \\ K(x_1,y) & K(x_1,x_1) \end{vmatrix}$$

$$= \frac{e^{ik|x-y|}}{2i}\sqrt{V(x)V(y)} - \frac{1}{k}\int_{-\infty}^{\infty} dx_1 \quad \begin{vmatrix} \frac{e^{ik|x-y|}}{2i}\sqrt{V(x)V(y)} & \frac{e^{ik|x-x_1|}}{2i}\sqrt{V(x)V(x_1)} \\ \frac{e^{ik|x_1-y|}}{2i}\sqrt{V(x_1)V(y)} & \frac{1}{2i}\,|V(x_1)| \end{vmatrix}$$

$$= \frac{e^{ik|x-y|}}{2i}\sqrt{V(x)V(y)} + \frac{1}{4k}\left[e^{ik|x-y|}\sqrt{V(x)V(y)} \int_{-\infty}^{\infty} dx_1 \; |V(x_1)| \right.$$

$$- \sqrt{V(x)V(y)} \int_{-\infty}^{\infty} dx_1 \quad e^{ik|x-x_1| + ik|x_1-y|} \quad |V(x_1)| \quad \Big]$$

$$= \sqrt{V(x)V(y)} \left\{ \frac{e^{ik|x-y|}}{2i} + \frac{1}{4k} \left[e^{ik|x-y|} \int_{-\infty}^{\infty} dx_1 \cdot |V(x_1)| - \int_{-\infty}^{\infty} dx_1 \; e^{ik|x-x_1| + ik|x_1-y|} |V(x_1)| \right] \right\}$$

$$= \sqrt{V(x)V(y)} \cdot \frac{e^{ik|x-y|}}{2i} \left\{ 1 - \frac{1}{2ik} \int_{-\infty}^{\infty} dx_1 \left[1 - e^{ik(|x-x_1| + |x_1-y| - |x-y|)} \right] \cdot |V(x_1)| \right\},$$

while

$$D(\lambda) \sim 1 - \frac{1}{2ik} \int_{-\infty}^{\infty} dx_1 \cdot |V(x_1)|$$

One can thus calculate approximately the resolvent kernel $H(x,y;\lambda)$ as

$$H(x,y;\lambda \overset{=\frac{1}{k}}{)} = \frac{N(x,y;\lambda)}{D(\lambda)} \sim \sqrt{V(x)V(y)} \; \frac{e^{ik|x-y|}}{2i} \; \frac{1 - \frac{1}{2ik} \int_{-\infty}^{\infty} dx_1 \left[1 - e^{ik(|x-x_1| + |x_1-y| - |x-y|)} \right] |V(x_1)|}{1 - \frac{1}{2ik} \int_{-\infty}^{\infty} dx_1 \; |V(x_1)|} .$$

Then the solution to the integral equation is given by

$$\phi(x) = e^{ikx} \sqrt{V(x)} + \frac{1}{k} \int_{-\infty}^{\infty} dy \; H(x,y;\tfrac{1}{k}) \; e^{iky} \sqrt{V(y)}$$

or,

$$\psi(x) \sim e^{ikx} + \int_{-\infty}^{\infty} dy \; \frac{e^{ik|x-y|}}{2ik} \; \frac{1 - \frac{1}{2ik} \int_{-\infty}^{\infty} dx_1 \left[1 - e^{ik(|x-x_1| + |x_1-y| - |x-y|)} \right] \cdot |V(x_1)|}{1 - \frac{1}{2ik} \int_{-\infty}^{\infty} dx_1 \cdot |V(x_1)|} \; V(y) \, e^{iky},$$

which is the "improved" scattering amplitude for $k \to \pm\infty$.

(14) $j(\varphi) = e^{ika\sin\phi} - \alpha \int_0^{2\pi} \frac{d\phi'}{2\pi} J_0\left(2ka \sin\frac{\phi-\phi'}{2}\right) \cdot j(\phi')$, $0 \le \phi < 2\pi$.

$\alpha \geq 0$.

(a) The kernel of this equation is translationally invariant <u>and</u> periodic. Hence, the eigenfunctions of the homogeneous equation

$$j(\phi) = -\alpha \int_0^{2\pi} \frac{d\phi'}{2\pi} J_0\left(2ka \sin\frac{\phi-\phi'}{2}\right) \cdot j(\phi') ,$$

are $e^{in\phi}$, where n: integer ($n = -\infty, \dots, -1, 0, 1, \dots, +\infty$).

The corresponding eigenvalues $-\alpha_n$ are equal to $-\alpha_n = \frac{1}{k_n}$ where k_n are the coefficients of the Fourier series of the kernel:

$$J_0\left(2ka\sin\frac{\phi}{2}\right) = \sum_{n=-\infty}^{\infty} k_n e^{in\phi} .$$

So, it suffices to determine the numbers k_n.

From the given formula for $J_n(x)$, it follows that $(n=0,\ x = 2ka\sin\frac{\phi}{2})$

$$J_0\left(2ka\sin\frac{\phi}{2}\right) = \frac{1}{2\pi}\int_0^{2\pi} d\xi\ e^{i2ka\sin\frac{\phi}{2}\sin\xi} = \frac{1}{2\pi}\int_0^{2\pi} d\xi\ e^{i2ka\sin\frac{\phi}{2}\cdot\cos\xi} ,$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\xi\ e^{i2ka\sin\frac{\phi}{2}\cdot\cos(-\xi)} = \frac{1}{2\pi}\int_0^{2\pi} d\xi\ e^{i2ka\sin\frac{\phi}{2}\cdot\cos\left(\frac{\phi}{2}-\xi\right)} ,$$

by taking $\xi \to \xi - \phi/2$ and keeping the range of integration intact since the integrand is periodic in ξ with period 2π. Furthermore,

$$2\sin\frac{\phi}{2}\cdot\cos\left(\frac{\phi}{2}-\xi\right) = \sin(\phi-\xi) + \sin\xi .$$

Hence,

$$J_0(2ka\sin\tfrac{\phi}{2}) = \int_0^{2\pi} \frac{d\xi}{2\pi} \; e^{ika\sin(\phi-\xi)} \cdot e^{ika\sin\xi} ,$$

i.e. $J_0(2ka\sin\frac{\phi}{2})$ is the autoconvolution (i.e., convolution with itself) of the function $e^{ika\sin\phi}$. From the given integral,

$$J_n(x) = \int_0^{2\pi} \frac{d\varphi'}{2\pi} \; e^{ix\sin\varphi'} \; e^{-in\varphi'} ,$$

it follows that $e^{ix\sin\phi}$ is expanded in Fourier series with coefficients $J_n(x)$:

$$e^{ix\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(x)\; e^{in\phi} \Rightarrow e^{ika\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(ka)\; e^{in\phi}.$$

Therefore, the Fourier series of $J_0(2ka\sin\frac{\phi}{2})$ has coefficients equal to $\left(\text{coefficients of } e^{+ika\sin\phi}\right)^2 = J_n(ka)^2$:

$$J_0(2ka\sin\phi/2) = \sum_{n=-\infty}^{\infty} \underbrace{J_n(ka)^2}_{k_n} \; e^{in\phi}.$$

It follows that the homogeneous equation has eigenvalues $-\alpha_n = \dfrac{1}{J_n(ka)^2} = k_n^{-1}$.

(b) It suffices to expand $e^{ika\sin\phi}$ in Fourier series. From (a) above, we get

$$e^{ika\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(ka)\; e^{in\phi} = \sum_{n=-\infty}^{\infty} f_n\, e^{in\phi}, \quad f_n \equiv J_n(ka).$$

By taking $j(\phi) = \sum_{n=-\infty}^{\infty} j_n\, e^{in\phi}$, one is led to the equation

$$j_n = f_n - \alpha k_n j_n \longrightarrow j_n = \frac{f_n}{1+\alpha k_n} = \frac{J_n(ka)}{1+\alpha J_n(ka)^2}$$

This solution is unique since $\alpha \geq 0$: $\alpha \neq$ eigenvalues.] So: $j(\varphi) = \sum_{n=-\infty}^{\infty} \dfrac{J_n(ka)}{1+\alpha J_n(ka)^2}\, e^{in\varphi}$.