18.307: Integral Equations **M.I.T. Department of Mathematics**

Spring 2006

Homework 8 **Due:** Wednesday, 04/26/2006

24. (Prob. 7.3 in text by M. Masujima.) Solve the boundary-value problem

$$\begin{cases} \psi_{xx} + \psi_{yy} + k^2\psi = 0, & (x, y) \in R, \\ \psi_y(x, 0) = e^{i\alpha x}, & x \geq 0, \\ \sqrt{x^2 + y^2}(\psi_y + ik\psi) \to 0, & \sqrt{x^2 + y^2} \to \infty, \end{cases}$$

by using the Wiener-Hopf technique. In the above, $\psi = \psi(x, y)$, R is the entire (x, y)-plane with the exception of the half-line $\{(x, y) : y = 0, x \geq 0\}$, k is real and positive, and $0 < \alpha < k$. **Note:** The given condition for $\sqrt{x^2 + y^2} \to \infty$ is called the "Sommerfeld radiation condition;" beware of the sign in this condition!

25. (From Example 7.4 in text by M. Masujima.) Solve the homogeneous integral equation

$$u(x) = \lambda \int_0^\infty dy \, \frac{u(y)}{\cosh\left[\frac{1}{2}(x - y)\right]}, \quad x \geq 0,$$

for the cases (a) $2\pi\lambda > 1$, and (b) $\lambda < 0$.

26. (Prob. 7.10 in text by M. Masujima.) For $0 < \lambda \leq 1$, solve the integral equation

$$u(x) = -\frac{\lambda}{2} \int_0^\infty dy \, \mathrm{Ei}(|x - y|) \, u(y), \quad x \geq 0,$$

where $\mathrm{Ei}(z)$ is the exponential integral defined as

$$\mathrm{Ei}(z) = -\int_z^\infty dt \, \frac{e^{-t}}{t}$$

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27. **Bonus problem:** (You don't lose any points if you don't solve this problem; you get extra credit if you work on it a bit.) Solve the Wiener-Hopf sum equation

$$\sum_{m=0}^{\infty} c_{n-m} \, u_m = \gamma^n, \quad n = 0, 1, 2, \ldots,$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, e^{-in\theta} \sqrt{\frac{(1 - \alpha e^{i\theta})(1 - \beta e^{-i\theta})}{1 - \alpha e^{-i\theta})(1 - \beta e^{i\theta})}}, \quad n = 0, \pm 1, \pm 2, \ldots,$$

$0 < \alpha < \beta < 1$ and $0 < \gamma < 1$; u_m are unknown.