18.307: Integral Equations

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Homework 7

Due: Wednesday, 04/19/06

21. This problem provides a review of contour integration and Fourier transforms in the complex plane. Consider the analytic function $f(z)$ of the complex variable z which, in particular, is differentiable in the interval (a, b) of the real axis. Define the functions

$$F(\zeta) = \int_a^b dx\, e^{-i\zeta x}\, f(x), \qquad I(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\zeta\, e^{i\zeta z}\, F(\zeta).$$

By breaking the integration path for $I(z)$ into the two halves $(-\infty, 0)$ and $(0, \infty)$, inserting the definition of $F(\zeta)$, and replacing the path (a, b) by either an upper or lower half-plane integration path, as appropriate, show that $I(z) = f(z)$ for $a < z < b$. What do you obtain for $I(z)$ when z is outside (a, b)?

22. (Prob. 7.21 in text by M. Masujima.) Solve the following Wiener-Hopf integral equation of the 1st kind by applying the Wiener-Hopf method:

$$\int_0^{\infty} dy\, K_0(x - y)\, u(y) = 2\pi, \quad x \geq 0,$$

where $K_0(x)$ is the modified Hankel function given by the integral formula

$$K_0(x) = \int_{-\infty}^{\infty} dt\, e^{-itx}\, (1 + t^2)^{-1/2}.$$

Remark: A version of this integral equation describes the problem of a viscous fluid past a semi-infinite plate; but you don't need to know this fact in order to solve the problem!

23. Solve the pair of the integral equations

$$\int_{-\infty}^{\infty} \frac{d\zeta}{2\pi}\, e^{i\zeta x} \mathcal{K}(\zeta)\, F(\zeta) = f(x), \quad x > 0,$$

$$\int_{-\infty}^{\infty} \frac{d\zeta}{2\pi}\, e^{i\zeta x} F(\zeta) = g(x), \quad x < 0,$$

that have to be satisfied *simultaneously*, where $\mathcal{K}(\zeta), f(x)$ and $g(x)$ are known and $F(\zeta)$ is unknown.