18.307: Integral Equations **M.I.T. Department of Mathematics**

Spring 2006

Homework 5 **Due:** Wednesday, 04/05/06

15. (Probs. 4.3 & 4.4 in text by M. Masujima.) By using Fourier transform, solve the integral equation

$$u(x) = f(x) + \lambda \int_{-\infty}^{\infty} dy\, e^{-|x-y|}\, u(y), \qquad -\infty < x < \infty,$$

for the following cases: (a) $f(x) = x$, $x > 0$; 0, $x < 0$, and (b) $f(x) = x$, $-\infty < x < \infty$.
Hint: In (a) you must define the Fourier transform $\tilde{f}(k)$ of $f(x)$ in the suitable part of the complex plane, so that the integral for $\tilde{f}(k)$ converges. In (b), you may write $f(x) = f_1(x) + f_2(x)$ where $f_i(x)$ ($i = 1, 2$) is zero for either $x < 0$ or $x > 0$, use the solution of part (a), and then superimpose to get the final solution.

16. (Prob. 5.9 in text by M. Masujima.) By using the bilinear formula for a symmetric kernel (which was given in class) show that, if $\check{\lambda}$ is an eigenvalue of the symmetric kernel $K(x, y)$, then the integral equation

$$u(x) = f(x) + \check{\lambda} \int_a^b dy\, K(x, y) u(y), \quad a \le x \le b,$$

has no solution, unless $f(x)$ is orthogonal to all the eigenfunctions corresponding to $\check{\lambda}$.

17. Consider the Fredholm integral equation of the 2nd kind

$$u(x) = f(x) + \lambda \int_0^1 dy\, \min\{x, y\}\, u(y),$$

where $\min\{x, y\}$ denotes the smallest of x and y.
(a) Find all non-trivial solutions $u_n(x)$ and corresponding eigenvalues λ_n for $f \equiv 0$.
Hint: Obtain a differential equation for $u(x)$ with the suitable conditions for $u(x)$ and $u'(x)$.
(b) For the original inhomogeneous equation ($f \neq 0$), will the iteration series converge? Explain.
(c) Evaluate the series $\sum_n \lambda_n^{-2}$ by using an appropriate integral.