18.307: Integral Equations **M.I.T. Department of Mathematics**

Spring 2006

Homework 4 **Due:** Wednesday, 03/22/06

11. (Similar to Prob. 4.1 in text by M. Masujima.) Show the following correspondence between the kernel $K(x, y)$ of the Fredholm equation and the determinant $D(\lambda)$ defined in class. What are the kernel eigenvalues in each case? Explain.
(a) $K(x, y) = \pm 1, \quad x \in [0,\, 1] \quad \Rightarrow \quad D(\lambda) = 1 \mp \lambda$.
(b) $K(x, y) = g(x)\, g(y), \quad x \in [a,\, b] \quad \Rightarrow \quad D(\lambda) = 1 - \lambda \int_a^b dx\, g(x)^2$.
(c) $K(x, y) = x + y, \quad x \in [0,\, 1] \quad \Rightarrow \quad D(\lambda) = 1 - \lambda - \frac{\lambda^2}{12}$.
(d) $K(x, y) = x^2 + y^2, \quad x \in [0,\, 1] \quad \Rightarrow \quad D(\lambda) = 1 - \frac{2}{3}\,\lambda - \frac{4}{45}\,\lambda^2$.
(e) $K(x, y) = xy(x + y), \quad x \in [0,\, 1] \quad \Rightarrow D(\lambda) = 1 - \frac{\lambda}{2} - \frac{1}{240}\,\lambda^2$.

12. Consider the Fredholm equation of the second kind
$$u(x) = f(x) + \lambda \int_a^b dx'\, K(x, x')\, u(x'), \qquad a \le x \le b.$$
(a) For $b = +\infty$, make the changes of variable $t = \frac{x}{1+x}$ and $t' = \frac{x'}{1+x'}$, which in turn renders the integration range finite. Write the original equation in terms of t and t'.
(b) Symmetrize the resulting kernel "as much as possible" by defining $(1 - t)\,(1 - t')\,\kappa(t,\, t') \equiv K(x,\, x')$. Show then that $\|\kappa\| = \|K\|$ and that the norm of the new inhomogeneous term also remains the same.

13. Consider the integral equation for the scattering of a non-relativistic electron by a potential,
$$\psi(x) = e^{ikx} + \int_{-\infty}^{\infty} dy\, \frac{e^{ik|x-y|}}{2ik}\, V(y)\, \psi(y), \qquad -\infty < x < \infty.$$
Symmetrize the kernel and find the first 2 terms of the Taylor series for the functions $D(\lambda)$ and $N(x, y; \lambda)$ defined in class. The ratio of these two series yields the improved Born series of the scattering amplitude ψ. Calculate this amplitude.

14. (Prob. 4.17 in text by M. Masujima.) In the theoretical search for "supergain antennas," maximizing the directivity in the far field of axially invariant currents $j = j(\phi)$ that flow along the surface of infinitely long, circular cylinders of radius a leads to the following Fredholm equation for the (unknown) density j:
$$j(\phi) = e^{ika \sin\phi} - \alpha \int_0^{2\pi} \frac{d\phi'}{2\pi}\, J_0\!\left(2ka\, \sin\frac{\phi - \phi'}{2}\right) j(\phi'), \qquad 0 \le \phi < 2\pi;$$
ϕ is the polar angle of the circular cross section, k is a positive constant proportional to frequency, α is a parameter (Lagrange multiplier) that expresses a constraint on the current magnitude, $\underline{\alpha \ge 0}$, and $J_0(x)$ is the Bessel function of zeroth order.
(a) Determine the eigenvalues of the homogeneous equation.
(b) Solve the given inhomogeneous equation in terms of Fourier series.
Hints for (a), (b): Use the integral formula $J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} d\phi'\, e^{ix \sin\phi'}\, e^{-in\phi'}$, n: integer and J_n: Bessel function of nth order.