18.307: Integral Equations | **M.I.T. Department of Mathematics**

Spring 2006

Homework 3 | **Due:** Wednesday, 03/15/06

7. (Similar to Prob. 3.9 in text by M. Masujima.) Solve the integral equation $\int_0^1 dy\, K(x,y)\, u(y) = 1$, where

$$K(x,y) = \begin{cases} xy + (x-y)^{-1/2}, & x > y \\ xy, & x < y. \end{cases}$$

8. (From Prob. 3.5 in text by M. Masujima.) Consider the integral equation

$$u(x) = f(x) + \lambda \int_x^{\infty} dy\, e^{a(x-y)}\, u(y), \quad a > 0,\ -\infty < x < \infty. \qquad (1)$$

(a) Is the kernel square integrable? Explain.
(b) Consider the homogeneous counterpart of (1), i.e., set $f \equiv 0$, and determine the λ's for which the resulting equation has non-trivial solutions, if there are any. Is the kernel spectrum (i.e., this set of λ's) discrete or continuous?
(c) Solve Eq. (1) for $f(x) = 1$. How many arbitrary constants does the solution have when $\lambda > 0$? How about $\lambda < 0$?
(d) Consider the integral equation stemming from (1) by replacing the kernel by its transpose, and find the non-trivial solutions $w(x)$ and the corresponding λ's.

9. (Prob. 2.11 in text by M. Masujima.) In solid-state physics, the effect of periodic forces in crystals on the steady-state motion of electrons is usually described by the time-independent Schrödinger equation with the periodic potential $V(x) = -(a^2 + k^2 \cos^2 x)$:

$$\frac{d^2\psi(x)}{dx^2} + (a^2 + k^2 \cos^2 x)\psi(x) = 0.$$

Show directly that even periodic solutions of this equation, which are called even Mathieu functions, satisfy the homogeneous integral equation

$$\psi(x) = \lambda \int_{-\pi}^{\pi} dy\, e^{k \cos x\, \cos y}\, \psi(y).$$

10. Antennas fed by transmission lines are often modeled by tubular dipoles with a current $I(x)$ that satisfies the Hallén integral equation:

$$\int_{-h}^{h} dy\, K(x-y)\, I(y) = A \sin(k|x|) + C\, \cos(kx), \quad |x| < h,$$

where A and C are constants, h is the length of the dipole and $k > 0$ is proportional to frequency. The kernel $K(x)$ is a known yet complicated function. In order to apply numerical methods to this equation, the exact kernel K is sometimes replaced by the simpler (approximate) kernel

$$K_{\mathrm{ap}}(x) = \frac{1}{4\pi} \frac{e^{ik\sqrt{x^2+a^2}}}{\sqrt{x^2+a^2}},$$

where a is the radius of the dipole tube. Give an argument to show that, with this approximate kernel, the equation for $I(x)$ has no solution.