18.303 Problem Set 5 Solutions

Problem 1: ((3+2+5+5)+(5+5) points)

(a) Solutions:

(i) $f(x)$ is bounded in every interval except intervals containing $x = 0$, so local integrability is trivial except for intervals containing $x = 0$. It is sufficient to consider integrals $\int_0^b$, because any interval $[a, b]$ containing 0 can be broken up into $\int_a^0 + \int_0^b$, and $f(-x) = f(x)$ so we only need to show that the latter is finite. But we can now just do the integral explicitly:

$$\int_0^b \ln x \, dx = \lim_{\epsilon \to 0^+} \int_\epsilon^b \ln x \, dx = \lim_{\epsilon \to 0^+} x \ln x - x|_\epsilon^b = b \ln b - b,$$

since

$$\lim_{\epsilon \to 0^+} x \ln x = 0,$$

as can easily be seen e.g. by L'Hôpital's rule applied to $\lim \frac{\ln x}{1/x} = \lim \frac{1/x}{-1/x^2} = \lim x = 0$.

(ii) $g(x)$ is not locally integrable for intervals containing the origin. For example

$$\int_0^b |g(x)| dx = \lim_{\epsilon \to 0^+} \int_\epsilon^b \frac{dx}{x} = \lim_{\epsilon \to 0^+} \ln x|_\epsilon^b = \infty.$$

Therefore, $f'\{\phi\} = f\{-\phi'\}$ is a singular distribution.

(iii) We write $f'\{\phi\} = \lim_{\epsilon \to 0^+} f'_\epsilon\{\phi\}$, and integrate by parts in $f'_\epsilon\{\phi\} = f_\epsilon\{-\phi'\}$:

$$\begin{aligned}
f'\{\phi\} &= -\lim_{\epsilon \to 0^+} \left[\int_{-\infty}^{-\epsilon} \underbrace{\ln(-x)}_{u} \underbrace{\phi'(x)dx}_{dv} + \int_\epsilon^\infty \underbrace{\ln(x)}_{u} \underbrace{\phi'(x)dx}_{dv} \right] \\
&= -\lim_{\epsilon \to 0^+} \left[\ln(\epsilon)\phi(-\epsilon) - \int_{-\infty}^{-\epsilon} \frac{-1}{x}\phi(x)dx - \ln(\epsilon)\phi(\epsilon) - \int_\epsilon^\infty \frac{1}{x}\phi(x)dx \right] \\
&= \lim_{\epsilon \to 0^+} \left[\int_{-\infty}^{-\epsilon} \frac{1}{x}\phi(x)dx + \int_\epsilon^\infty \frac{1}{x}\phi(x)dx \right] + \cancel{\lim_{\epsilon \to 0^+} [\ln(\epsilon) (\phi(\epsilon) - \phi(-\epsilon))]}.
\end{aligned}$$

In the last line, the first limit is precisely the Cauchy Principal Value of $\int g(x)\phi(x)dx$ (CPV = remove a ball of radius ϵ around the singularity, do the integral, and then take the $\epsilon \to 0^+$ limit). The second term vanishes because, since $\phi(x)$ is continuous and infinitely differentiable, $\phi(\epsilon) - \phi(-\epsilon)$ vanishes at least as fast as ϵ as $\epsilon \to 0$, so its product with $\ln \epsilon$ vanishes in the limit as in part (i).[1]

(iv) Use $f'\{\phi\} = f\{-\phi'\} = f\{-[\phi - \phi(0)]'\} = f'\{\phi - \phi(0)\}$. Then substitute $\phi - \phi(0)$ into the previous part, and note that since the integrand $\frac{\phi(x)-\phi(0)}{x}$ is now finite as $x \to 0$ [since $\phi(x) - \phi(0)$ goes to zero as $x \to 0$, at least proportionally to x or faster as in the previous part], we can now just take the limit to write $f'\{\phi\} = \int_{-\infty}^{\infty} g(x)[\phi(x) - \phi(0)]dx$ without using the CPV.

(b) Solutions:

(i) Let V^c denote the complement of V (the exterior region outside V, i.e. $V^c = \mathbb{R}^d \backslash V$), and note that

[1] This last fact is a little subtle. Naively, we could just write the Taylor expansion $\phi(x) = \phi(0) + \phi'(0)\epsilon + O(\epsilon^2)$ to obtain $\phi(\epsilon) - \phi(-\epsilon) = 2\phi'(0)\epsilon + O(\epsilon^2)$. However, it turns out that test functions $\phi(x)$ are not analytic (do not have a convergent Taylor series) at all x. Nevertheless, because ϕ is differentiable with bounded ϕ', it follows that $\phi(x)$ is "Lipshitz continuous:" $|\phi(x) - \phi(y)| < K|x - y|$ for all x, y and for some constant $K > 0$ (e.g. you can prove this from the mean-value theorem of analysis), so $\phi(\epsilon) - \phi(-\epsilon) < 2K\epsilon$ and the result follows. But I don't expect you to provide this level of proof; a Taylor argument is acceptable.

$\partial V^c = \partial V$ (but with the outward-normal vector reversed in sign). We write:

$$\begin{aligned}
\nabla f\{\phi\} = f\{-\nabla\phi\} &= -\int_V f_1 \nabla\phi - \int_{V^c} f_2 \nabla\phi \\
&= -\int_V [\nabla(f_1\phi) - \phi\nabla f_1] - \int_{V^c} [\nabla(f_2\phi) - \phi\nabla f_2] \\
&= -\oint_{\partial V} f_1\phi\mathbf{n} + \int_V \phi\nabla f_1 + \oint_{\partial V} f_2\phi\mathbf{n} + \int_{V^c} \phi\nabla f_2 \\
&= \left[\delta(\partial V)\,[f_2(\mathbf{x}) - f_1(\mathbf{x})]\,\mathbf{n}(\mathbf{x}) + \begin{cases} \nabla f_1(\mathbf{x}) & \mathbf{x} \in V \\ \nabla f_2(\mathbf{x}) & \mathbf{x} \notin V \end{cases}\right] \{\phi\}
\end{aligned}$$

as desired.

(ii) In this case, we will need to integrate by parts twice, but we can just quote the results from the "notes on elliptic operators" from class (where we integrated by parts twice with $-\nabla^2$ already), albeit keeping the boundary terms from ∂V that were zero in the notes:

$$\begin{aligned}
\nabla^2 f\{\phi\} = f\{\nabla^2\phi\} &= \int_V f_1\nabla^2\phi + \int_{V^c} f_2\nabla^2\phi \\
&= \oint_{\partial V} [(f_1 - f_2)\nabla\phi - \phi(\nabla f_1 - \nabla f_2)]\cdot\mathbf{n} + \int_V \phi\nabla^2 f_1 + \int_{V^c} \phi\nabla^2 f_2.
\end{aligned}$$

But the first term is $\delta(\partial V)\,[f_1(\mathbf{x}) - f_2(\mathbf{x})]\,\{\mathbf{n}\cdot\nabla\phi\} = (\mathbf{n}\cdot\nabla)\delta(\partial V)\,[f_2(\mathbf{x}) - f_1(\mathbf{x})]\,\{\phi\}$ by the definition (note the sign change) of the distributional derivative $\mathbf{n}\cdot\nabla$ (note that this is a scalar derivative in the $\mathbf{n}$ direction, not a gradient vector). The second term is a surface delta function weighted by $(\mathbf{n}\cdot\nabla f_1 - \mathbf{n}\cdot\nabla f_2)$,the discontinuity in the normal derivative. And the last terms are just a regular distribution. So, we have

$$\nabla^2 f = (\mathbf{n}\cdot\nabla)\delta(\partial V)\,[f_2 - f_1] + \delta(\partial V)\,[\mathbf{n}\cdot\nabla f_1 - \mathbf{n}\cdot\nabla f_2] + \begin{cases} \nabla^2 f_1(\mathbf{x}) & \mathbf{x} \in V \\ \nabla^2 f_2(\mathbf{x}) & \mathbf{x} \notin V \end{cases}.$$

As noted in class, $\mathbf{n}\cdot\nabla$ of a delta function is a "dipole" oriented in the $\mathbf{n}$ direction, so the first term is a "dipole layer".

Problem 2: ((5+10)+2 points)

(a) We solve for $g(r)$ in 3d as follows:

(i) For $r > 0$, $-\nabla^2 g - \omega^2 g = 0$ and hence $-\omega^2 g = \nabla^2 g = \frac{1}{r}(rg)'' \implies h'' = -\omega^2 h$ where $h(r) = rg(r)$. The solution to this is $h(r) = ce^{i\omega r} + de^{-i\omega r}$ for some constants c and d, or $g(r) = \frac{ce^{i\omega r} + de^{-i\omega r}}{r}$

It is a little more tricky to determine whether we should use the c or the d term than in class, since both decay at the same rate. The ratio c/d will be determined by some kind of boundary condition at infinity, but what might this be? It is acceptable for you to just punt on this here; since $e^{\pm i\omega r}$ are complex conjugates of each other, your analysis will apply equally well to either one, and you can arbitrarily pick one, say $ce^{i\omega r}/r$, to analyze.

However, to see why there should be a sensible choice, recall that this operator arose in pset 5 by assuming a time dependence $e^{-i\omega t}$ multiplying the solution, in which case we are looking at wave solutions $\frac{ce^{i\omega(r-t)} + de^{-i\omega(r+t)}}{r}$, where the c term describes waves moving *out* towards $r \to \infty$, while the d term describes waves moving *in* from infinity. In wave problems, we typically impose a boundary condition of *outgoing waves at infinity*, in which case we must set $d = 0$. (However, the choice would have been reversed if we picked the opposite sign convention, $e^{+i\omega t}$, for the time dependence.)

(ii) Let's focus on $g(r) = ce^{i\omega r}/r$. As in class, the $1/r$ singularity is no problem in 3d (it is cancelled by the

Jacobian factor $r^2 dr$), so g is a regular distribution. Given an arbitrary test function $q(\mathbf{x})$, we now evaluate

$$\begin{aligned}
(\hat{A}g)\{q\} = g\{\hat{A}q\} &= \int g\hat{A}q \\
&= \int_0^\infty r^2 dr \int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta \left[-\omega^2 gq - \frac{g}{r}\frac{\partial^2}{\partial r^2}(rq) + \underbrace{\cancel{(\theta, \phi \text{ derivatives of } q)}}_{\text{integrate to 0, from class}} \right] \\
&= \iint \sin\theta\, d\theta\, d\phi \left[\int_0^\infty c \left(-\omega^2 rqe^{i\omega r} - \underbrace{e^{i\omega r}\frac{\partial^2}{\partial r^2}[rq]}_{\text{int. by parts}} \right) dr \right] \\
&= \iint \sin\theta\, d\theta\, d\phi \left[-ce^{i\omega r}\frac{\partial}{\partial r}[rq]\Big|_0^\infty + c\int_0^\infty \left(-\omega^2 rqe^{i\omega r} + \underbrace{i\omega e^{i\omega r}\frac{\partial}{\partial r}[rq]}_{\text{int. by parts}} \right) dr \right] \\
&= \iint \sin\theta\, d\theta\, d\phi \left[cq(0) - \cancel{ic\omega e^{i\omega r}[rq]\Big|_0^\infty} + c\int_0^\infty \cancel{\left(-\omega^2 rqe^{i\omega r} + \omega^2 e^{i\omega r}[rq] \right)} dr \right] \\
&= 4\pi cq(0),
\end{aligned}$$

and hence $\boxed{c = 1/4\pi}$. Thus

$$G(\mathbf{x}, \mathbf{x}') = \frac{e^{i\omega|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}$$

assuming boundary conditions such that $d = 0$. More generally, since exactly the same result applies to the $de^{-i\omega r}/r$ term, we obtain,

$$G(\mathbf{x}, \mathbf{x}') = \frac{ce^{i\omega|\mathbf{x}-\mathbf{x}'|} + de^{-i\omega|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$$

for $\boxed{c + d = 1/4\pi}$, with the ratio c/d being set by the boundary conditions at ∞. The value at $\mathbf{x} = \mathbf{x}'$ being irrelevant in the distribution sense, e.g. we can assign it to zero, since this is a regular distribution with a finite integral, similar to class.)

(b) The $\omega \to 0$ limit gives $1/4\pi|\mathbf{x} - \mathbf{x}'|$ as in class, by inspection.

18.303 Linear Partial Differential Equations: Analysis and Numerics
Fall 2014