

## 18.303 Problem Set 4 Solutions

### Problem 1: (10+10)

- (a)  $\hat{A}$  is self-adjoint under the inner product  $\langle u, v \rangle = \int_{\Omega} \bar{u}v/c$ , from class (and the 1d version in homework), and hence the eigenfunctions are orthogonal for distinct eigenvalues (but not necessarily normalized to  $\|u_n\| = 1$ ), hence  $\alpha_n = \langle u_n, f \rangle / \langle u_n, u_n \rangle$ . The expression  $u(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}') d^2 \mathbf{x}'$  is just the solution to  $\hat{A}u = f$ , i.e. it is  $u = \hat{A}^{-1}f = \sum_n \frac{\alpha_n}{\lambda_n} u_n(\mathbf{x})$ .
- (b) This is just  $\langle u, \hat{A}^{-1}u \rangle / \langle u, u \rangle$ , which is a Rayleigh quotient for the self-adjoint operator  $\hat{A}^{-1}$ , and hence (thanks to the min-max theorem) it is bounded above by the **largest** eigenvalue of  $\hat{A}^{-1}$ , which is  $1/\lambda_1$  (the inverse of the **smallest** eigenvalue of  $\hat{A}$ ). (Note that by positive-definiteness  $0 < \lambda_1 < \lambda_2 < \dots$ )

### Problem 2:

In this problem, we will solve the Laplacian eigenproblem  $-\nabla^2 u = \lambda u$  in a 2d radius-1 cylinder  $r \leq 1$  with Dirichlet boundary conditions  $u|_{r=1\Omega} = 0$  by “brute force” in Julia with a 2d finite-difference discretization, and compare to the analytical Bessel solutions. You will find the IJulia notebooks posted on the 18.303 website for Lecture 9 and Lecture 11 extremely useful! (Note: when you open the notebook, you can choose “Run All” from the Cell menu to load all the commands in it.)

- (a) Using the notebook for a  $100 \times 100$  grid, compute the 6 smallest-magnitude eigenvalues and eigenfunctions of  $A$  with `λi, Ui=eigs(Ai,nev=6,which='SM')`. The eigenvalues are given by `λi`. The notebook also shows how to compute the exact eigenvalue from the square of the root of the Bessel function. Compared with the high-accuracy  $\lambda_1$  value, compute the error  $\Delta\lambda_1$  in the corresponding finite-difference eigenvalue from the previous part. Also compute  $\Delta\lambda_1$  for  $N_x = N_y = 200$  and 400. How fast is the convergence rate with  $\Delta x$ ? Can you explain your results, in light of the fact that the center-difference approximation we are using has an error that is supposed to be  $\sim \Delta x^2$ ? (Hint: think about how accurately the boundary condition on  $\partial\Omega$  is described in this finite-difference approximation.)
- (b) Modify the above code to instead discretize  $\nabla \cdot c\nabla$ , by writing  $A_0$  as  $-G^T C_g G$  for some  $G$  matrix that implements  $\nabla$  and for some  $C_g$  matrix that multiplies the gradient by  $c(r) = r^2 + 1$ . Draw a sketch of the grid points at which the components of  $\nabla$  are discretized—these will *not* be the same as the  $(n_x, n_y)$  where  $u$  is discretized, because of the centered differences. Be careful that you need to evaluate  $c$  at the  $\nabla$  grid points now! Hint: you can make the matrix  $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$  in Julia by the syntax `[M1;M2]`.

Hint: Notice in the IJulia notebook from Lecture 11 how a matrix `r` is created from a column-vector of `x` values and a row-vector of `y` values. You will need to modify these `x` and/or `y` values to evaluate `r` on a new grid(s). Given the `r` matrix `rc` on this new grid, you can evaluate  $c(r)$  on the grid by `c = rc.^2 + 1`, and then make a diagonal sparse matrix of these values by `spdiags(reshape(c, prod(size(c))))`.

- (c) Using this  $A \approx \nabla \cdot c\nabla$ , compute the smallest- $|\lambda|$  eigensolution and plot it. Given the eigenfunction converted to a 2d  $N_x \times N_y$  array `u`, as in the Lecture 11 notebook, plot  $u(r)$  as a function of  $r$ , along with a plot of the exact Bessel eigenfunction  $J_0(k_0 r)$  from the  $c = 1$  case for comparison.

```
plot(r[Nx/2:end,Ny/2], u[Nx/2:end,Ny/2])
k0 = so.newton(x -> besselj(0,x), 2.0)
plot(0:0.01:1, besselj(0, k0 * (0:0.01:1))/50)
```

Here, I scaled  $J_0(k_0 r)$  by 1/50, but you should change this scale factor as needed to make the plots of comparable magnitudes. Note also that the  $r$  array here is the radius evaluated on the original  $u$  grid, as in the Lecture 11 notebook.

Can you qualitatively explain the differences?

### Problem 3: (5+5+10+10+5 points)

Recall that the displacement  $u(x, t)$  of a stretched string [with fixed ends:  $u(0, t) = u(L, t) = 0$ ] satisfies the wave equation  $\frac{\partial^2 u}{\partial x^2} + f(x, t) = \frac{\partial^2 u}{\partial t^2}$ , where  $f(x, t)$  is an external force density (pressure) on the string.

- (a) Suppose that  $\tilde{u}$  solves  $\frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{f}(x, t) = \frac{\partial^2 \tilde{u}}{\partial t^2}$  and satisfies  $\tilde{u}(0, t) = \tilde{u}(L, t) = 0$ . Now, consider  $u = \Re \tilde{u} = \frac{\tilde{u} + \bar{\tilde{u}}}{2}$ . Clearly,  $u(0, t) = u(L, t) = 0$ , so  $u$  satisfies the same boundary conditions. It also satisfies the PDE:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{1}{2} \left[ \frac{\partial^2 \tilde{u}}{\partial t^2} + \overline{\frac{\partial^2 \tilde{u}}{\partial t^2}} \right] \\ &= \frac{1}{2} \left[ \frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{f} + \overline{\frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{f}} \right] \\ &= \frac{\partial^2 u}{\partial x^2} + f, \end{aligned}$$

since  $f = \frac{\tilde{f} + \bar{\tilde{f}}}{2} = \Re \tilde{f}$ . The key factors that allowed us to do this are (i) *linearity*, and (ii) the *real-ness* of the PDE (the PDE itself contains no  $i$  factors or other complex coefficients).

- (b) Plugging  $u(x, t) = v(x)e^{-i\omega t}$  and  $f(x, t) = g(x)e^{-i\omega t}$  into the PDE, we obtain

$$\frac{\partial^2 v}{\partial x^2} e^{-i\omega t} + g e^{-i\omega t} = -\omega^2 v e^{-i\omega t},$$

and hence

$$\left( -\frac{\partial^2}{\partial x^2} - \omega^2 \right) v = g$$

and  $\hat{A} = -\frac{\partial^2}{\partial x^2} - \omega^2$ . The boundary conditions are  $v(0) = v(L) = 0$ , from the boundary conditions on  $u$ .

Since  $\omega^2$  is real, this is in the general Sturm-Liouville form that we showed in class is **self-adjoint**.

Subtracting a constant from an operator just shifts all of the eigenvalues by that constant, keeping the eigenfunctions the same. Thus  $\hat{A}$  is still positive-definite if  $\omega^2$  is  $<$  the smallest eigenvalue of  $-\partial^2/\partial x^2$ , and positive semidefinite if  $\omega^2 =$  the smallest eigenvalue. In this case, we know analytically that the eigenvalues of  $-\partial^2/\partial x^2$  with these boundary conditions are  $(n\pi/L)^2$  for  $n = 1, 2, \dots$ . So  $\hat{A}$  is positive-definite if  $\omega^2 < (\pi/L)^2$ , it is positive-semidefinite if  $\omega^2 = (\pi/L)^2$ , and it is indefinite otherwise.

- (c) We know that  $\hat{A}G(x, x') = 0$  for  $x \neq x'$ . Also, just as in class and as in the notes, the fact that  $\hat{A}G(x, x') = \delta(x - x')$  means that  $G$  must be continuous (otherwise there would be a  $\delta'$  factor) and  $\partial G/\partial x$  must have a jump discontinuity:

$$\frac{\partial G}{\partial x} \Big|_{x=x'+} - \frac{\partial G}{\partial x} \Big|_{x=x'-} = -1$$

for  $-\partial^2 G/\partial x^2$  to give  $\delta(x - x')$ . We could also show this more explicitly by integrating:

$$\begin{aligned} \int_{x'-0^+}^{x'+0^+} \hat{A}G \, dx &= \int_{x'-0^+}^{x'+0^+} \delta(x - x') \, dx = 1 \\ &= -\frac{\partial G}{\partial x} \Big|_{x'-0^+}^{x'+0^+} - \int_{x'-0^+}^{x'+0^+} \omega^2 G \, dx, \end{aligned}$$

which gives the same result.

Now, similar to class, we will solve it separately for  $x < x'$  and for  $x > x'$ , and then impose the continuity requirements at  $x = x'$  to find the unknown coefficients.

For  $x < x'$ ,  $\hat{A}G = 0$  means that  $\frac{\partial^2 G}{\partial x^2} = -\omega^2 G$ , hence  $G(x, x')$  is some sine or cosine of  $\omega x$ . But since  $G(0, x') = 0$ , we must therefore have  $G(x, x') = \alpha \sin(\omega x)$  for some coefficient  $\alpha$ .

Similarly, for  $x > x'$ , we also have a sine or cosine of  $\omega x$ . To get  $G(L, x') = 0$ , the simplest way to do this is to use a sine with a phase shift:  $G(x, x') = \beta \sin(\omega[L - x])$  for some coefficient  $\beta$ .

Continuity now gives two equations in the two unknowns  $\alpha$  and  $\beta$ :

$$\begin{aligned} \alpha \sin(\omega x') &= \beta \sin(\omega[L - x']) \\ \alpha \omega \cos(\omega x') &= -\beta \omega \cos(\omega[L - x']) + 1, \end{aligned}$$

which has the solution  $\alpha = \frac{1}{\omega \cos(\omega x') \sin(\omega[L - x']) + \sin(\omega x') \cos(\omega[L - x'])}$ ,  $\beta = \frac{\sin(\omega x')}{\omega \cos(\omega x') \sin(\omega[L - x']) + \sin(\omega x') \cos(\omega[L - x'])}$ . This simplifies a bit from the identity  $\sin A \cos B + \cos B \sin A = \sin(A + B)$ , and hence

$$G(x, x') = \frac{1}{\omega \sin(\omega L)} \begin{cases} \sin(\omega x) \sin(\omega[L - x']) & x < x' \\ \sin(\omega x') \sin(\omega[L - x]) & x \geq x' \end{cases}$$

which obviously obeys reciprocity.

Note that if  $\omega$  is an eigenfrequency, i.e.  $\omega = n\pi/L$  for some  $n$ , then this  $G$  blows up. The reason is that  $\hat{A}$  in that case is singular (the  $n$ -th eigenvalue was shifted to zero), and defining  $\hat{A}^{-1}$  is more problematical. (Physically, this corresponds to driving the oscillating string at a resonance frequency, which generally leads to a diverging solution unless there is dissipation in the system.)

- (d) It is critical to get the signs right. Recall that we approximated  $\frac{d^2}{dx^2}$  by  $-D^T D$  for a 1st-derivative matrix  $D$ . Therefore, you want to make a matrix  $A = D^T D - \omega^2 I$ . In Matlab,  $A = D' * D - \omega^2 * \text{eye}(N)$  where  $N$  is the size of the matrix  $D = \text{diff1}(N) / dx$  with  $dx = 1/(N+1)$ . We make  $\mathbf{d}_k$  by  $\mathbf{d}_k = \text{zeros}(N, 1)$ ;  $\mathbf{d}_k(k) = 1/dx$ . I used  $N = 100$ .

The resulting plots are shown in figure 1, for  $\omega = 0.4\pi/L$  (left) and  $\omega = 5.4\pi/L$  (right) and  $x' = 0.5$  and  $0.75$ . As expected, for  $\omega < \pi/L$  where it is positive-definite, the Green's function is positive and qualitatively similar to that for  $\omega = 0$ , except that it is slightly curved. For larger  $\omega$ , though, it becomes oscillatory and much more interesting in shape. The exact  $G$  and the finite-difference  $G$  match very well, as they should, although the match becomes worse at higher frequencies—the difference approximations become less and less accurate as the function becomes more and more oscillatory relative to the grid  $\Delta x$ , just as was the case

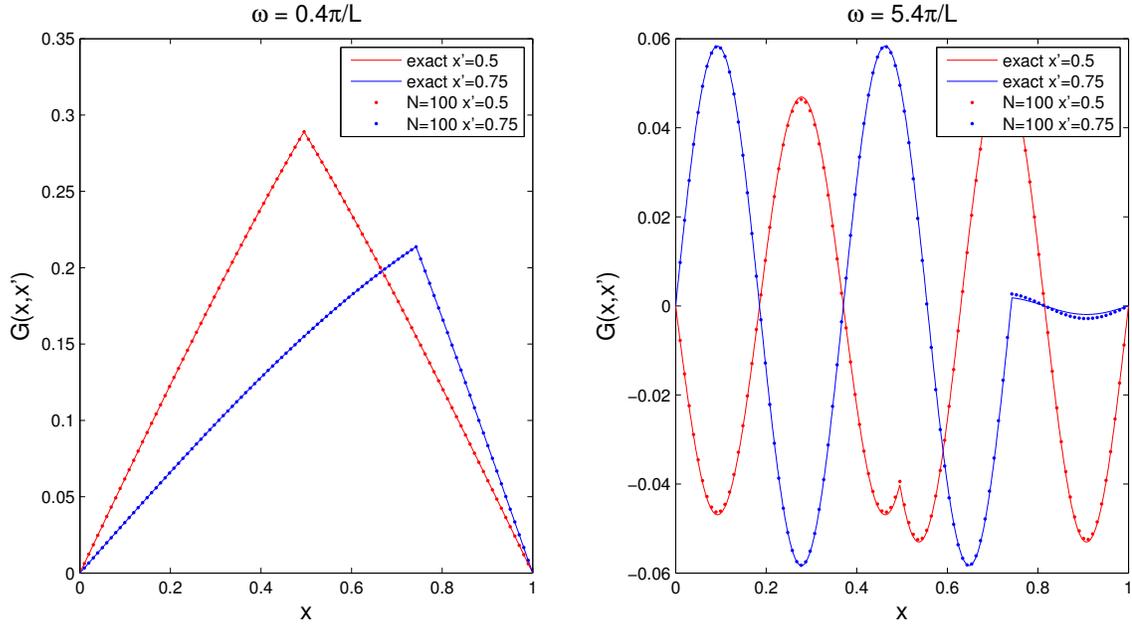


Figure 1: Exact (lines) and finite-difference (dots) Green's functions  $G(x, x')$  for  $\hat{A} = -\frac{\partial^2}{\partial x^2} - \omega^2$ , for two different  $\omega$  values (left and right) and two different  $x'$  values (red and blue).

for the eigenfunctions as discussed in class.

[If you try to plug in an eigenfrequency (e.g.  $4\pi/L$ ) for  $\omega$ , you may notice that the exact  $G$  blows up but the finite-difference  $G$  is finite. The reason for this is simple: the eigenvalues of the finite-difference matrix  $D^T D$  are not exactly  $(n\pi/L)^2$ , so the matrix is not exactly singular, nor is it realistically possible to make it exactly singular thanks to rounding errors. If you plug in something very close to an eigenfrequency, then  $G$  becomes very close to an eigenfunction multiplied by a large amplitude. It is easy to see why this is if you look at the expansion of the solution in terms of the eigenfunctions.]

(e) For small  $\omega$ ,  $\sin(\omega y) \approx \omega y$  for any  $y$ , and hence

$$G(x, x') \rightarrow \frac{1}{\omega^2 L} \begin{cases} \omega^2 x(L - x') & x < x' \\ \omega^2 x'(L - x) & x \geq x' \end{cases},$$

which matches the Green's function of  $-\frac{d^2}{dx^2}$  from class. (Equivalently, you get  $0/0$  as  $\omega \rightarrow 0$  and hence must use L'Hôpital's rule or similar to take the limit.)

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18.303 Linear Partial Differential Equations: Analysis and Numerics  
Fall 2014

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