Universal randomness in 2D

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Abstract

We begin by reviewing one-dimensional random objects that are *universal* in the sense that they arise in many contexts – in particular as scaling limits of large families of discrete models – and *canonical* in the sense that they are uniquely characterized by scale invariance and other natural symmetries. Examples include Brownian motion, Bessel processes, stable Lévy processes and ranges of stable subordinators.

We then introduce several universal and canonical random objects that are (at least in some sense) two dimensional or planar, along with discrete analogs of these objects. These include trees, distributions, curves, loop ensembles, surfaces, and growth trajectories. Keywords include continuum random tree, stable Lévy tree, stable looptree, Gaussian free field, Schramm-Loewner evolution, percolation, uniform spanning tree, loop-erased random walk, Ising model, FK cluster model, conformal loop ensemble, Brownian loop soup, random planar map, Liouville quantum gravity, Brownian map, Brownian snake, diffusion limited aggregation, first passage percolation, and dielectric breakdown model.

Finally, we discuss the intricate and surprising relationships *between* these universal objects. We explain how to use generalized functions to construct curves and vice versa; conformally weld a pair of surfaces to produce a surface decorated by a simple curve; topologically and conformally mate pairs of trees to obtain surfaces decorated by non-simple curves; and *reshuffle* these constructions to describe random growth trajectories on random surfaces. We present both discrete and continuum analogs of these relationships. Keywords include imaginary geometry, quantum zipper, peanosphere, and quantum Loewner evolution.

Contents

Preface

The goals of this book are very simple. We will

1. introduce a few fundamental random objects, and
2. explain how they are related to one another.

The fundamental random objects include processes, trees, distributions (a.k.a. generalized functions), curves, loop ensembles, surfaces, and growth trajectories.

All of these objects are in some sense *universal*. That is, they arise as *macrosopic limits* of many different kinds of random systems, which may have very different *microscopic* behavior. This usage of the term "universal" comes from statistical physics. Physicists tell us that many phenomena (such as phase transitions) are surprisingly similar from one material to another. Physical systems — and mathematical models — that look very different on the microscopic level (different atoms, molecules, etc.) are declared to belong the same *universality class* if they behave the same way in some macroscopic limit. The convergence of general random walks to Brownian motion (under only a mild second moment condition) is an example of mathematical universality. We will encounter many other examples during the course of this book, some proven and some conjectural.

The random objects introduced in this book are also all in some sense *canonical*. Many fundamental objects in mathematics are singled out by special symmetries. For example, in a universe full of roughly round-ish shapes, the sphere stands out; it is uniquely determined by rotational invariance, equidistance of points from a center, etc. Similarly, among all random variables taking values in the space of continuous paths, Brownian motion is (up to multiplicative constant) the only one with reflection invariance, stationarity, and independence of increments. It has a strong claim to be *the* canonical continuous random path. This book will survey objects that can claim with equal justification to be the canonical random planar tree, the canonical random non-self-crossing curve, the canonical random surface, and so forth.

Among the various symmetries that make these objects special, many involve some sort of *conformal invariance*. Recall that the Riemann uniformization theorem implies the existence of a conformal map between any two sphere-homeomorphic surfaces; when the sphere is replaced by a multi-handled torus or a disk with holes, the space of conformal equivalence classes (a.k.a. the *moduli space*) remains finite dimensional. This remarkable fact is a peculiar feature of two dimensions and seems to be a large part of what makes the two dimensional theory interesting. In the 1980's and 1990's a branch of physics called *conformal field theory*, motivated by both string theory and two dimensional statistical mechanics, began to discover and explore some surprisingly far reaching consequences of conformal symmetry assumptions in physical models.

Mathematicians have more recently expanded these ideas further, building in particular on the introduction of the so-called Schramm-Loewner evolution in 1999.

The focus of this text is on the mathematics, and in particular on a few of the most fundamental discrete and continuum mathematical objects in one and two dimensions. However we will provide some cursory discussion of the motivating problems that link them to physics and to other fields.

The first half of this book introduces both discrete and continuum analogs of several universal random objects: processes, trees, distributions, curves, loop ensembles, surfaces, and growth trajectories. The second half explores the intricate and often surprising relationships between these objects. To put this another way, the first half of the book introduces a certain cast of characters, and the second half explores the drama that takes place when these characters interact.

This book is intended as a broad introductory overview of this field and as such it covers a good deal of material. With additional detail, each individual chapter could be (and in many cases already has been) expanded into an entire book of its own. We do not provide fully detailed proofs of every result cited in this text. However, we aim to provide enough rigor and detail to enable the reader to appreciate the overall narrative and to begin further research in this field.

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1 Random processes

1.1 Brownian motion

We recall the basic construction, Itô's formula, martingale representation theorem, local martingales, and quadratic variation. More detailed accounts of this material can be found in basic probability texts like [Dur10], the book on Brownian motion by Mörters and Peres [MP10], and the stochastic calculus texts [KS91, RY99].

1.2 Bessel processes

An introduction to Bessel process can be founded for example in [RY99, Chapter 11]. The idea is to construct a solution to the stochastic differential equation

$$dX_t = dB_t + \frac{\delta - 1}{2} X_t^{-1} dt,$$

where B_t is standard Brownian motion and δ is a fixed real constant. The interesting question is how to extend this definition beyond times at which X_t reaches zero. One approach is to have the process jump up by ϵ each time it hits 0, and take a limit as $\epsilon \to 0$. Another is to define the square of the process Bessel process (which turns out to fit more neatly into the framework of some general theorems in SDE theory, and allows us to show that the process is adapted to the Brownian motion). A simple application of Itô's formula allows us to check which power of X_t (depending on δ) is a martingale, and also to prove that summing independnet δ_1 and δ_2 Bessel processes produces a $\delta_1 + \delta_2$ Bessel process.

1.3 Brownian excursions, meanders, and bridges

One may define a Brownian excursion indexed by $[0, 1]$ by conditioning a Brownian motion, started at ϵ, to end in $[0, \epsilon]$, and then taking the $\epsilon \to 0$ limit. Brownian motion conditioned to stay in a cone (starting from the apex) is explained in [Shi85] along with the relationship to Bessel processes.

1.4 Stable Lévy processes

We recall Lamperti's classic work on continuous state branching processes [Lam67] and textbooks on Lé'vy processes by Sato, by Bertoin and by Barndoff-Nielson, Mikosch, and Resnick [Sat99, Ber96, BNMR01]

1.5 Ranges of stable subordinators

The range of a stable subordinator is a random closed subset of $\mathbf{R}_+$. it can be understood as the zero set of a Bessel process. If we condition the endpoints of the Bessel process to be zero at 1, we can also define a random closed subset of $[0, 1]$. These random sets can be characterized by renewal and scale invariance properties, which are similar to the properties we will later use to characterize conformal loop ensembles (the complement of the union of the interiors of these loops will turn out to be a random subset of $\mathbf{R}^2$).

2 Random trees

2.1 Galton-Watson trees

Galton-Watson trees and their scaling limits are described by Duquesne and Le Gall in [DLG05]. See also [LGLJ98, DLG06, DLG09]. One of the interesting features of

Galton-Watson trees is the phase transition: when the expected number of children is less than one, the tree is easily seen to be finite almost surely. (The expected number of children at level k decays exponentially in k.) When the expected number of children is greater than one, the tree has a positive probability of being infinite.

When the expected number of children is equal to 1, one may observe offspring sets of vertices one at a time, exploring tree boundary in a clockwise way, so that the number of live vertices is a martingale. This martingale is closely related to the contour function of the tree (but not exactly the same; see Lévy tree story below).

2.2 Aldous's continuum random tree

The continuum random tree was introduced in a series of papers by Aldous in 1991 [Ald91a, Ald91b, Ald93]. It can be understood as a scaling limit of Galton-Watson trees.

2.3 Lévy trees and stable looptrees

There are some very simple analogs of the CRT in which stable Lévy excursions play the role of the Brownian excursion [DLG05]. These can also be understood as scaling limits of Galton-Watson trees, when the number of children has a power law tail (finite mean but infinite variance).

There is a closely related construction in which each of the countably many big branch points is replaced with a loop; the resulting "tree of loops" called a looptree. See the work by Curien and Korchemski on *stable looptrees* [CK13], as well as the exposition in [DMS14].

2.4 Brownian snakes

A Brownian snake is essentially a Brownian motion indexed by a CRT. It will play a role later in the construction of a certain canonical random surface called the Brownian map, but it was actually studied independently before its relationship to random surfaces was discovered [DLG05].

3 Random generalized functions

3.1 Tempered distributions and Fourier transforms

The *Schwartz space* on $\mathbb{R}^d$ is the space of C^∞ functions ϕ such that for any multi-indices α and β in which each of the seminorms $\sup D^\alpha \phi(x) x^\beta$ is bounded. These seminorms induce a topology on the Schwartz space; continuous linear functionals on the Schwartz space are called *tempered distributions*. The space of tempered distributions is the smallest space which includes the bounded continuous functions and is closed under both differentiation and the Fourier transform.

In the exposition on Gaussian free fields, we will often find it convenient to limit attention to compactly supported test functions (instead of test functions in the Schwartz space) as this will allow us to more easily isolate the effects of boundary conditions.

3.2 Gaussian free fields

Gaussian Hilbert spaces are introduced in [Jan97]. Surveys of the Gaussian free field can be found in [She07, Ber].

3.3 Fractional Gaussian fields and log correlated free fields

The GFF can be generalized in several ways. See the survey articles [DRSV14b, LSSW14] for more on fractional Gaussian fields and log correlated Gaussian fields in general d-dimensional spaces. These are obtained by applying powers of the Laplacian to white noise. The Gaussian free field can be understood as the restriction to two dimensions of log correlated fields defined in higher dimensions.

3.4 Dimer models and uniform spanning trees

The UST height function is arguably the simplest discrete analog of the GFF. See Kenyon's scaling limit proof [Ken00b, Ken01], which makes use of the equivalent formulation of the model in terms of dimers.

4 Random curves and loop ensembles

4.1 Schramm-Loewner evolution: basic definitions and phases

Much of the work on Schramm-Loewner evolution is prefigured in the physics literature on conformal field theory [DFMS97]. Schramm's original paper [Sch00] has been followed by many excellent survey articles and textbooks [Wer03, KN04, Car06, Law09, BN11]. The so-called natural parameterization is described in [LS11, LR12, LZ13].

4.2 Loop erased random walk and uniform spanning tree

See Wilson's algorithm [Wil96, PW98] and the original UST/LERW convergence paper [LSW04].

4.3 Critical percolation interfaces

Percolation interface scaling limits are tractable thanks to a fundamental discovery by Stanislav Smirnov [Smi01].

4.4 Gaussian free field level lines

See [SS05, SS09, SS13] and the universality theorem in [Mil10].

4.5 Ising, Potts, and FK-cluster models

Some of these "next simplest after percolation" models are also tractable [CS]

4.6 Bipolar orientations

This is another simple model conjectured to scale to SLE_{12}. The conjecture is easy to state, but the motivation behind the conjecture will not be explained until the sections on imaginary geometry and the peanosphere.

4.7 Restriction measures, self-avoiding walk, and loop soups

The relationship between $\mathrm{SLE}_{8/3}$ and Brownian motion is especially beautiful and has an especially beautiful history. See the account in the early work by Lawler, Schramm, and Werner [LSW03].

4.8 Conformal loop ensembles

Given that the discrete interfaces that scale to SLE have "loop ensemble" variants, one would expect there to be a natural "loop ensemble" variant of SLE itself. See the introduction in [She09, SW12].

5 Random surfaces

5.1 Planar maps

A planar map is a planar map together with an embedding in the plane (defined up to topological equivalence). Enumeration work was done by Tutte in the 1960's [Tut62, Tut68].

5.2 Decorated surfaces and Laplacian determinants

The Laplacian determinant and its inverse are related to partition functions for the GFF and UST models in surprisingly simple ways. See Kenyon's work on scaling limits of determinant Laplacians on grids [Ken00a] and the broad survey by Merris [Mer94] which describes Kirchhoff's matrix tree theorem, among other things.

Given any finite connected graph (V, E) the Laplacian on the graph can be defined as a linear operator Δ from $\mathbf{R}^V$ itself. Its matrix is given by

$$M_{i,j} = \begin{cases} 1 & i \neq j, (v_i, v_j) \in E \\ 0 & i \neq j, (v_i, v_j) \notin E \\ -\deg(v_i) & i = j. \end{cases}.$$

Let $R \subset \mathbf{R}^V$ be the set of functions with mean zero. Then $-\Delta : R \to R$ is invertible, and Kirchhoff's matrix tree theorem states that if α is the determinant of this invertible operator on R then α is the number of spanning trees of V. The quantity α is also the product of all of the non-zero eigenvalues of the matrix M.

The DGFF partition function can be be written $\int_R (2\pi)^{-|V-1|/2} e^{-(f,-\Delta f)/2} df$. Expanding over eigenbases, and using the fact the $\frac{1}{\sqrt{2\pi}} \int e^{-tx^2/2} dt = t^{-1/2}$, we find that quantity is $\alpha^{-1/2}$.

5.3 Mullin-Bernardi bijection

There is a very simple bijection between discrete lattice walks in $\mathbb{Z}_+^2$ starting and ending at zero and rooted planar maps with distinguished spanning trees. See [Mul67, Ber07] as well as the exposition in [She11].

5.4 Cori-Vaquelin-Schaeffer bijection

The Cori-Vaquelin-Schaeffer bijection gives a way to bijectively count *undecorated* planar maps [CV81, JS98, Sch99]. Every quadrangulation with a root can be decorated by a directed breadth first spanning tree spannign all of the edges. When multiple incoming edges come into the same vertex, each outgoing edge is connected to only one of them in this tree namely, the next one over in clockwise ordering.

5.5 Hamburger-cheeseburger bijection

There is a generalization of the Mullin-Bernardi bijection in which the rooted planar map comes with an arbitrary distinguished edge subset, instead of a distinguished spanning tree [She11].

5.6 Bipolar bijection

The scaling limit of the pair of trees can be easily described in this case, as it can in each of the other cases described above.

5.7 Brownian map

The idea behind the Cori-Vaquelin-Schaeffer bijection can be used to define a continuum random metric space [MM06, LG13, Mie13, LG14], which has a natural infinite volume analog [CL12]. See Le Gall's ICM notes [Le 14] or the survey by Miermont and Le Gall [LGM+12]. An axiomatic characterization of the Brownian map in terms of it symmetries appears in [MS15a].

5.8 Liouville quantum gravity

Polaykov conceived of a random surface model based on an action closely related to the Gaussian free field [Pol81]. If h is an instance of the Gaussian free field, one attempts to define a meausre of the form $e^{\gamma h(z)} dz$, which in turn encodes the volume form of a random surface after a conformal map back to a fixed parameter space (say, a disk in the plane). The rigorous construction of this random measure was given by Høegh-Krohn in 1971 [HK71], for the range $\gamma \in [0, \sqrt{2})$, and the full range $[0, 2)$ was treated by Kahane (who used the term *multiplicative chaos*) in 1985 [Kah85], see also the survey [RV14]. The construction of the measure as a measure-valued function on the space of instances h of the GFF was done in [DS11]. The case $\gamma = 2$ is different but one can make sense of the measure by different means [DRSV14a, DRSV14c].

5.9 KPZ (Knizhnik-Polyakov-Zamolodchikov) scaling relations

A relationship between scaling dimensions was discovered by Knizhnik, Polyakov, and Zamolodchikov in [KPZ88]. In a recent memoir [Pol08] Polyakov explains how the discover of this relationship cemented the belief that the discrete planar map models were (in some sense) equivalent to Liouville quantum gravity. See [DS11] and the references therein. See the Hausdorff variant in [RV08].

See the derivation of the $d = 26$ value for the bosonic string by Lovelace in 1971 in [Lov71].

5.10 Quantum wedges, cones, spheres and disks

There are natural ways to define quantum surfaces using Bessel process excursions. There are some natural probability measures on the space of infinite volume surfaces. There are also some natural infinite measures on the space of finite volume surfaces. See the introduction to [DMS14].

6 Random growth trajectories

6.1 Eden model and first passage percolation

There are a number of natural growth trajectories. The Eden model, introduced by Edein in 1961 [Ede61], is the simplest to describe. Here, every edge has an exponential clock, and when it rings the edge is added to the growing edge cluster (if it is incident to the existing cluster). A generalization of this story known as first-passage percolation was introduced by Hammersley and Welsh in 1965 [HW65].

6.2 Diffusion limited aggregation and the dieelectric breakdown model

Diffusion limited aggregation, as introduced by Witten and Sander in 1981 [WJS81, WS83], is a model for growth in which new particle locations on the boundary are chosen from harmonic measure instead of uniform measure. See early conjectures in [Mea86] and the theorem of Kesten [Kes87].

6.3 KPZ (Kardar-Parisi-Zhang) growth

The KPZ growth model is the logarithm of the stochastic heat equation with geometric noise. It was introduced in a slightly different form, and without a rigorous construction, Karder, Parisi, and Zhang in [KPZ86]. It does not itself describe the conjectural scaling limit of Eden model fluctuations; rather, it describes what amounts to a sort of "off critical" variant, which is believed to converge to a fixed point as a certain parameter tends to zero. These models can be viewed as interesting in their right, or interesting as approximations to the (still conjectural) KPZ fixed point, which is in turn the conjectural scaling limit of Eden model fluctuations. The fixed point conjecture is described by Corwin and Quastel in [CQ11]. See also Corwin's survey article [Cor12].

6.4 Hastings-Levitov

The Hastings-Levitov model was designed as an approximation of what should be a continuum DLA theory. The hope was that one could prove the existence of an isotropic scaling limits of this model, and that would be easier than establishing the analogous result for (an isotropic form of) ordinary DLA. While this goal has not yet been achieved, there has been some recent progress in understanding Hastings-Levitov; see, e.g., [JVST12].

6.5 Internal DLA

Internal DLA is a growth model introduced by Meakin and Deuthch in 1986 [MD86]. Internal DLA growth seems to be much smoother than Eden model, with logarithmic fluctuations [LBG92, JLS12, JLS13, AG13b, AG13a]. Unlike ordinary (external) DLA and most of the other growth models presented in this section, fluctuations of internal DLA on the grid have a well understood scaling limit, which can be described by a variant of the Gaussian free field [JLS+14].

7 Imaginary geometry

7.1 Flow lines starting from the boundary

The results in this section are detailed in a series of *imaginary geometry* papers by the current authors [MS12a, MS12b, MS12c, MS13a]. The idea is to try to define flow lines of $e^{ih(z)/\chi}$ where $\chi > 0$ is a fixed parameter and h is an instance of the Gaussian free field. We begin by discussing paths that originate at the boundary, and are related to forms of chordal SLE.

7.2 Interior flow lines

It is similarly possible to make sense of flow lines of $e^{ih(z)/\chi}$ starting from interior points of a planar domain.

7.3 Counterflow lines and space-filling SLE

The tree and dual tree of flow lines have an interface that can be described as a space-filling curve.

7.4 Time reversal symmetries

Imaginary geometry can be used to prove several basic facts about SLE, including time reversal symmetry for several forms of SLE_κ with $\kappa < 4$ and $\mathrm{SLE}_{\kappa'}$ with $\kappa' > 4$.

8 Conformal welding and the quantum zipper

8.1 Welding simple quantum wedges

One can "conformally weld" two quantum wedges to each other to obtain a new thicker quantum wedge. The first version of this story (which applies to two wedges of a particular thickness) was described in [She10]

8.2 Welding more general quantum wedges

Additional welding constructions are described in [DMS14]. These allow one to weld together two wedges of weights W_1 and W_2 to produce a new wedge of weight $W_1 + W_2$. One can also weld the left and right sides of a single quantum wedge to each other, to produce a quantum cone.

9 Mating trees and the peanosphere

9.1 Moore's theorem and topological tree mating

There is a simple way to see that gluing two continuum random trees produces a topological sphere decorated by a space-filling path [DMS14].

9.2 Matings from complex dynamics

The idea of topologically mating Julia sets is given an overview in [Mil04, Mil06] and the references therein.

9.3 Matings of correlated continuum random trees

The peanosphere theorem is proved in [DMS14].

9.4 Matings of trees of disks

The "tree of disk" analog of the peanosphere theorem is also proved in [DMS14].

9.5 Relation to discrete bijections

The tree mating construction has discrete analogs in the planar maps decorated by FK models, uniform spanning trees, and bipolar orientations, as detailed in the various bijections described in Section 5.

10 Quantum Loewner evolution

10.1 Reshuffling in discrete examples

Discrete and continuum analogs of the quantum Loewner evolution are introduced in [MS13b]. We begin with an account of the discrete models.

The recurrence of random walk on the infinite tree decorated map is a special case of the result in [GGN13].

DLA and the Eden model (and more generally DBM) can be defined on random planar maps. They are related, respectively, to loop erased random walk and a Bernoulli percolation interface via a certain "reshuffling" construction.

In the case of the Eden model, some relevant work on exploring triangulations by Angel and Schramm appears in [Ang03, AS03]

10.2 Defining QLE

This reshuffling has a continuum analog that can be shown to converge (at least subsequentially) [MS13b].

10.3 QLE and the Brownian map

This continuum exploration introduced in [MS13b] can be used to prove the equivalence of $\sqrt{8/3}$-Liouville quantum gravity and the Brownian map. This is accomplished in a series of papers [MS15b, MS15c, MS15d].

It remains an open problem to endow γ-LQG with a metric space structure for general γ. A famous calculation of Watabiki describes what can be conjectured to be the Hausdorff dimension of general γ-LQG surfaces [Wat93].

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