18.175: Lecture 5

More integration and expectation

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Outline

Integration

Expectation

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Expectation

Recall Lebesgue integration

- Lebesgue: If you can measure, you can integrate.
- In more words: if $(\Omega, \mathcal{F})$ is a measure space with a measure μ with $\mu(\Omega) < \infty$) and $f : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable, then we can define $\int f d\mu$ (for non-negative f, also if both $f \vee 0$ and $-f \wedge 0$ and have finite integrals...)
- Idea: define integral, verify linearity and positivity (a.e. non-negative functions have non-negative integrals) in 4 cases:
  - f takes only finitely many values.
  - f is bounded (hint: reduce to previous case by rounding down or up to nearest multiple of ϵ for $\epsilon \to 0$).
  - f is non-negative (hint: reduce to previous case by taking $f \wedge N$ for $N \to \infty$).
  - f is any measurable function (hint: treat positive/negative parts separately, difference makes sense if both integrals finite).

Lebesgue integration

- **Theorem:** if f and g are integrable then:
  - If $f \geq 0$ a.s. then $\int f d\mu \geq 0$.
  - For $a, b \in \mathbb{R}$, have $\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu$.
  - If $g \leq f$ a.s. then $\int g d\mu \leq \int f d\mu$.
  - If $g = f$ a.e. then $\int g d\mu = \int f d\mu$.
  - $|\int f d\mu| \leq \int |f| d\mu$.
- When $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}^d, \mathcal{R}^d, \lambda)$, write $\int_E f(x)dx = \int 1_E f d\lambda$.

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- Given probability space $(\Omega, \mathcal{F}, P)$ and random variable X, we write $EX = \int XdP$. Always defined if $X \geq 0$, or if integrals of $\max\{X, 0\}$ and $\min\{X, 0\}$ are separately finite.
- EX^k is called k**th moment of** X. Also, if $m = EX$ then $E(X - m)^2$ is called the **variance** of X.

Properties of expectation/integration

- **Jensen's inequality:** If μ is probability measure and $\phi : \mathbb{R} \to \mathbb{R}$ is convex then $\phi(\int f d\mu) \leq \int \phi(f) d\mu$. If X is random variable then $E\phi(X) \geq \phi(EX)$.
- **Main idea of proof:** Approximate ϕ below by linear function L that agrees with ϕ at EX.
- **Applications:** Utility, hedge fund payout functions.
- **Hölder's inequality:** Write $\|f\|_p = (\int |f|^p d\mu)^{1/p}$ for $1 \leq p < \infty$. If $1/p + 1/q = 1$, then $\int |fg| d\mu \leq \|f\|_p \|g\|_q$.
- **Main idea of proof:** Rescale so that $\|f\|_p \|g\|_q = 1$. Use some basic calculus to check that for any positive x and y we have $xy \leq x^p/p + y^q/p$. Write $x = |f|$, $y = |g|$ and integrate to get $\int |fg| d\mu \leq \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p \|g\|_q$.
- **Cauchy-Schwarz inequality:** Special case $p = q = 2$. Gives $\int |fg| d\mu \leq \|f\|_2 \|g\|_2$. Says that dot product of two vectors is at most product of vector lengths.

Bounded convergence theorem

- **Bounded convergence theorem:** Consider *probability* measure μ and suppose $|f_n| \leq M$ a.s. for all n and some fixed $M > 0$, and that $f_n \to f$ in probability (i.e., $\lim_{n\to\infty} \mu\{x : |f_n(x) - f(x)| > \epsilon\} = 0$ for all $\epsilon > 0$). Then

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

  (Build counterexample for infinite measure space using wide and short rectangles?...)
- **Main idea of proof:** for any ϵ, δ can take n large enough so $\int |f_n - f| d\mu < M\delta + \epsilon$.

- **Fatou's lemma:** If $f_n \geq 0$ then

$$\liminf_{n\to\infty} \quad f_n d\mu \geq \quad \big(\liminf_{n\to\infty} f_n\big) d\mu.$$

  (Counterexample for opposite-direction inequality using thin and tall rectangles?)

- **Main idea of proof:** first reduce to case that the f_n are increasing by writing $g_n(x) = \inf_{m\geq n} f_m(x)$ and observing that $g_n(x) \uparrow g(x) = \liminf_{n\to\infty} f_n(x)$. Then truncate, used bounded convergence, take limits.

More integral properties

- **Monotone convergence:** If $f_n \geq 0$ and $f_n \uparrow f$ then

$$\int f_n d\mu \uparrow \int f d\mu.$$

- **Main idea of proof:** one direction obvious, Fatou gives other.
- **Dominated convergence:** If $f_n \to f$ a.e. and $|f_n| \leq g$ for all n and g is integrable, then $\int f_n d\mu \to \int f d\mu$.
- **Main idea of proof:** Fatou for functions $g + f_n \geq 0$ gives one side. Fatou for $g - f_n \geq 0$ gives other.

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Spring 2014

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