18.175: Lecture 4

Integration

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Outline

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Integration

Expectation

Recall definitions

- **Probability space** is triple $(\Omega, \mathcal{F}, P)$ where Ω is sample space, $\mathcal{F}$ is set of events (the σ-algebra) and $P : \mathcal{F} \to [0, 1]$ is the probability function.
- σ-**algebra** is collection of subsets closed under complementation and countable unions. Call $(\Omega, \mathcal{F})$ a measure space.
- **Measure** is function $\mu : \mathcal{F} \to \mathbb{R}$ satisfying $\mu(A) \geq \mu(\emptyset) = 0$ for all $A \in \mathcal{F}$ and countable additivity: $\mu(\cup_i A_i) = \sum_i \mu(A_i)$ for disjoint A_i.
- Measure μ is **probability measure** if $\mu(\Omega) = 1$.
- The **Borel** σ-**algebra** $\mathcal{B}$ on a topological space is the smallest σ-algebra containing all open sets.

Recall definitions

- Real random variable is function $X : \Omega \to \mathbb{R}$ such that the preimage of every Borel set is in $\mathcal{F}$.
- Note: to prove X is measurable, it is enough to show that the pre-image of every open set is in $\mathcal{F}$.
- Can talk about σ-algebra generated by random variable(s): smallest σ-algebra that makes a random variable (or a collection of random variables) measurable.

Lebesgue integration

- Lebesgue: If you can measure, you can integrate.
- In more words: if $(\Omega, \mathcal{F})$ is a measure space with a measure μ with $\mu(\Omega) < \infty$) and $f : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable, then we can define $\int f d\mu$ (for non-negative f, also if both $f \vee 0$ and $-f \wedge 0$ and have finite integrals...)
- Idea: define integral, verify linearity and positivity (a.e. non-negative functions have non-negative integrals) in 4 cases:
  - f takes only finitely many values.
  - f is bounded (hint: reduce to previous case by rounding down or up to nearest multiple of ϵ for $\epsilon \to 0$).
  - f is non-negative (hint: reduce to previous case by taking $f \wedge N$ for $N \to \infty$).
  - f is any measurable function (hint: treat positive/negative parts separately, difference makes sense if both integrals finite).

Lebesgue integration

- Can we extend previous discussion to case $\mu(\Omega) = \infty$?
- **Theorem:** if f and g are integrable then:

  If $f \geq 0$ a.s. then $\int f d\mu \geq 0$.
  For $a, b \in \mathbb{R}$, have $\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu$.
  If $g \leq f$ a.s. then $\int g d\mu \leq \int f d\mu$.
  If $g = f$ a.e. then $\int g d\mu = \int f d\mu$.
  $|\int f d\mu| \leq \int |f| d\mu$.
- When $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}^d, \mathcal{R}^d, \lambda)$, write $\int_E f(x) dx = \int 1_E f d\lambda$.

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