18.175: Lecture 39

Last lecture

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Outline

Recollections

Strong Markov property

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Strong Markov property

More σ-algebra thoughts

- Write $\mathcal{F}_s^o = \sigma(B_r : r \leq s)$.
- Write $\mathcal{F}_s^+ = \cap_{t>s}\mathcal{F}_t^o$
- Note right continuity: $\cap_{t>s}\mathcal{F}_t^+ = \mathcal{F}_s^+$.
- $\mathcal{F}_s^+$ allows an "infinitesimal peek at future"

Looking ahead

- **Expectation equivalence theorem** If Z is bounded and measurable then for all $s \geq 0$ and $x \in \mathbb{R}^d$ have

$$E_x(Z|\mathcal{F}_s^+) = E_x(Z|\mathcal{F}_s^o).$$

- **Proof idea:** Consider case that $Z = \sum_{i=1}^m f_m(B(t_m))$ and the f_m are bounded and measurable. Kind of obvious in this case. Then use same measure theory as in Markov property proof to extend general Z.
- **Observe:** If $Z \in \mathcal{F}_s^+$ then $Z = E_x(Z|\mathcal{F}_s^o)$. Conclude that $\mathcal{F}_s^+$ and $\mathcal{F}_s^o$ agree up to null sets.

Blumenthal's 0-1 law

- If $A \in \mathcal{F}_0^+$, then $P(A) \in \{0, 1\}$ (if P is probability law for Brownian motion started at fixed value x at time 0).
- There's nothing you can learn from infinitesimal neighborhood of future.
- **Proof:** If we have $A \in \mathcal{F}_0^+$, then previous theorem implies

$$1_A = E_x(1_A | \mathcal{F}_0^+) = E_x(1_A | \mathcal{F}_0^o) = P_x(A) \quad P_x \mathrm{a.s.}$$

Markov property

- If $s \geq 0$ and Y is bounded and $\mathcal{C}$-measurable, then for all $x \in \mathbb{R}^d$, we have

$$E_x(Y \circ \theta_s | \mathcal{F}_s^+) = E_{B_s} Y,$$

  where the RHS is function $\phi(x) = E_x Y$ evaluated at $x = B_s$.
- **Proof idea:** First establish this for some simple functions Y (depending on finitely many time values) and then use measure theory (monotone class theorem) to extend to general case.

More observations

- If $\tau = \inf\{t \geq 0 : B_t > 0\}$ then $P_0(\tau = 0) = 1$.
- If $T_0 = \inf\{t > 0 : B_t = 0\}$ then $P_0(T_0 = 0) = 1$.
- If B_t is Brownian motion started at 0, then so is process defined by $X_0 = 0$ and $X_t = tB(1/t)$. (Proved by checking $E(X_s X_t) = stE(B(1/s)B(1/t)) = s$ when $s < t$. Then check continuity at zero.)

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Stopping time

- A random variable S taking values in $[0, \infty]$ is a **stopping time** if for all $t \geq 0$, we have $\{S > t\} \in \mathcal{F}_t$.
- Distinction between $\{S < t\}$ and $\{S \leq t\}$ doesn't make a difference for a right continuous filtration.
- Example: let $S = \inf\{t : B_t \in A\}$ for some open (or closed) set A.

Strong Markov property

- Let $(s,\omega) \to Y_s(\omega)$ be bounded and $\mathcal{R} \times \mathcal{C}$-measurable. If S is a stopping time, then for all $x \in \mathbb{R}^d$
$$E_x(Y_S \circ \theta_S | \mathcal{F}_S) = E_{B(S)} Y_S \text{ on } \{S < \infty\},$$
where RHS means function $\phi(x,t) = E_x Y_t$ evaluated at $x = B(S)$, and $t = S$.
- In fact, similar result holds for more general Markov processes (Feller processes).
- **Proof idea:** First consider the case that S a.s. belongs to an increasing countable sequence (e.g., S is a.s. a multiple of 2^{-n}). Then this essentially reduces to discrete Markov property proof. Then approximate a general stopping time by a discrete time by rounding down to multiple of 2^{-n}. Use some continuity estimates, bounded convergence, monotone class theorem to conclude.
- Extend optional stopping to continuous martingales similarly.

Continuous martingales

- **Question:** If B_t is a Brownian motion, then is $B_t^2 - t$ a martingale?
- **Question:** If B_t and $\tilde{B}_t$ are independent Brownian motions, then is $B_t \tilde{B}_t$ a martingale?
- **Question:** If B_t is a martingale, then is $e^{B_t - t/2}$ a martingale?
- **Question:** If B_t is a Brownian motion in $\mathbb{C}$ (i.e., real and imaginary parts are independent Brownian motions) and f is an analytic function on $\mathbb{C}$, is $f(B_t)$ a complex martingale?
- **Question:** If B_t is a Brownian motion on $\mathbb{R}^d$ and f is a harmonic function on $\mathbb{R}^d$, is $f(B_t)$ a martingale?
- **Question:** Suppose B_t is a one dimensional Brownian motion, and $g_t : \mathbb{C} \to \mathbb{C}$ is determined by solving the ODE

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - 2B_t}, \quad g_0(z) = z.$$

  Is $\arg(g_t(z) - W_t)$ a martingale?

Farewell... and for future reference..

- Course has reached finite stopping time. Process goes on.
- Future probability graduate courses include
  - 18.177: fall 2014 (Jason Miller)
  - 18.177: spring 2015 (Alice Giuonnet)
  - 18.176: fall or spring 2015-16
- Probability seminar: Mondays at 4:15.
- I am happy to help with quals and reading.
- Talk to friendly postdocs: Vadim Gorin, Jason Miller, Jonathon Novak, Charlie Smart, Nike Sun, Hao Wu.
- Thanks for taking the class!