18.175: Lecture 32

More Markov chains

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Outline

General setup and basic properties

Recurrence and transience

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Markov chains: general definition

- Consider a measurable space $(S, \mathcal{S})$.
- A function $p : S \times \mathcal{S} \to \mathbb{R}$ is a **transition probability** if
  - For each $x \in S$, $A \to p(x, A)$ is a probability measure on $S, \mathcal{S})$.
  - For each $A \in S$, the map $x \to p(x, A)$ is a measurable function.
- Say that X_n is a **Markov chain** w.r.t. $\mathcal{F}_n$ with transition probability p if $P(X_{n+1} \in B | \mathcal{F}_n) = p(X_n, B)$.
- How do we construct an infinite Markov chain? Choose p and initial distribution μ on $(S, \mathcal{S})$. For each $n < \infty$ write

$$P(X_j \in B_j, 0 \le j \le n) = \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots$$

$$\int_{B_n} p(x_{n-1}, dx_n).$$

  Extend to $n = \infty$ by Kolmogorov's extension theorem.

Markov chains

- **Definition, again:** Say X_n is a **Markov chain** w.r.t. $\mathcal{F}_n$ with transition probability p if $P(X_{n+1} \in B | \mathcal{F}_n) = p(X_n, B)$.
- **Construction, again:** Fix initial distribution μ on $(S, \mathcal{S})$. For each $n < \infty$ write

$$P(X_j \in B_j, 0 \le j \le n) = \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots$$

$$\int_{B_n} p(x_{n-1}, dx_n).$$

  Extend to $n = \infty$ by Kolmogorov's extension theorem.
- **Notation:** Extension produces probability measure P_μ on sequence space $(S^{0,1,\ldots}, \mathcal{S}^{0,1,\ldots})$.
- **Theorem:** $(X_0, X_1, \ldots)$ chosen from P_μ is Markov chain.
- **Theorem:** If X_n is any Markov chain with initial distribution μ and transition p, then finite dim. probabilities are as above.

Markov properties

- **Markov property:** Take $(\Omega_0, \mathcal{F}) = \left(S^{\{0,1,\ldots\}}, \mathcal{S}^{\{0,1,\ldots\}}\right)$, and let P_μ be Markov chain measure and θ_n the shift operator on Ω_0 (shifts sequence n units to left, discarding elements shifted off the edge). If $Y : \Omega_0 \to \mathbb{R}$ is bounded and measurable then
$$E_\mu(Y \circ \theta_n | \mathcal{F}_n) = E_{X_n} Y.$$
- **Strong Markov property:** Can replace n with a.s. finite stopping time N and function Y can vary with time. Suppose that for each n, $Y_n : \Omega_n \to \mathbb{R}$ is measurable and $|Y_n| \leq M$ for all n. Then
$$E_\mu(Y_N \circ \theta_N | \mathcal{F}_N) = E_{X_N} Y_N,$$
where RHS means $E_x Y_n$ evaluated at $x = X_n, n = N$.

Properties

- **Property of infinite opportunities:** Suppose X_n is Markov chain and

$$P(\cup_{m=n+1}^{\infty}\{X_m \in B_m\}|X_n) \geq \delta > 0$$

  on $\{X_n \in A_n\}$. Then $P(\{X_n \in A_n \, i.o.\} - \{X_n \in B_n \, i.o.\}) = 0$.
- **Reflection principle:** Symmetric random walks on $\mathbb{R}$. Have $P(\sup_{m \geq n} S_m > a) \leq 2P(S_n > a)$.
- **Proof idea:** Reflection picture.

Reversibility

- Measure μ called **reversible** if $\mu(x)p(x,y) = \mu(y)p(y,x)$ for all x, y.
- Reversibility implies stationarity. Implies that amount of mass moving from x to y is same as amount moving from y to x. Net flow of zero along each edge.
- Markov chain called reversible if admits a reversible probability measure.
- Are all random walks on (undirected) graphs reversible?
- What about directed graphs?

Cycle theorem

- **Kolmogorov's cycle theorem:** Suppose p is irreducible. Then exists reversible measure if and only if
  - $p(x,y) > 0$ implies $p(y,x) > 0$
  - for any loop $x_0, x_1, \ldots x_n$ with $\prod_{i=1}^n p(x_i, x_{i-1}) > 0$, we have

$$\prod_{i=1}^n \frac{p(x_{i-1}, x_i)}{p(x_i, x_{i-1})} = 1.$$

- Useful idea to have in mind when constructing Markov chains with given reversible distribution, as needed in Monte Carlo Markov Chains (MCMC) applications.

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Query

- **Interesting question:** If A is an infinite probability transition matrix on a countable state space, what does the (infinite) matrix $I + A + A^2 + A^3 + \ldots = (I - A)^{-1}$ represent (if the sum converges)?
- **Question:** Does it describe the expected number of y hits when starting at x? Is there a similar interpretation for other power series?
- How about e^A or $e^{\lambda A}$?
- Related to distribution after a Poisson random number of steps?

Recurrence

- Consider probability walk from y ever returns to y.
- If it's 1, return to y infinitely often, else don't. Call y a **recurrent state** if we return to y infinitely often.