18.175: Lecture 29

Still more martingales

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Martingales

- Let $\mathcal{F}_n$ be increasing sequence of σ-fields (called a **filtration**).
- A sequence X_n is **adapted** to $\mathcal{F}_n$ if $X_n \in \mathcal{F}_n$ for all n. If X_n is an adapted sequence (with $E|X_n| < \infty$) then it is called a **martingale** if
$$E(X_{n+1}|\mathcal{F}_n) = X_n$$
for all n. It's a **supermartingale** (resp., **submartingale**) if same thing holds with $=$ replaced by $\leq$ (resp., $\geq$).
- **Theorem:** $E(X|\mathcal{F}_i)$ is a martingale if $\mathcal{F}_i$ is an increasing sequence of σ-algebras and $E(|X|) < \infty$.
- **Optional stopping theorem:** Under some conditions (what conditions?) the expectation of martingale at a stopping time is just the initial value of martingale.
- **Martingale convergence:** A non-negative martingale almost surely has a limit. Under some conditions (what conditions?) the expectation of the limit is the initial value of the martingale.

Problems

- **Classic brainteaser:** 52 cards (half red, half black) shuffled and face down. I turn them over one at a time. At some point (before last card is turned over) you say "stop". If subsequent card is red, you get one dollar. You do you time your stop to maximize your probability of winning?
- **Classic observation:** if r_n denotes fraction of face-down cards that are red after n have been turned over, then r_n is a martingale.
- Optional stopping theorem implies that it doesn't matter when you say stop. All strategies yield same expected payoff.
- Odds of winning are same for monkey and genius.
- Unless you cheat.
- **Classic question:** Is this also true of the stock market?

Martingales as real-time subjective probability updates

- Ivan sees email from girlfriend with subject "some possibly serious news", thinks there's a 20 percent chance she'll dump him by email's end. Revises number after each line:
- Oh Ivan, I've missed you so much! 12
- But there's something I have to tell you 23
- and please don't take this the wrong way. 29
- I've been spending lots of time with a guy named Robert, 47
- a visiting database consultant on my project 34
- who seems very impressed by my work 23
- and wants me to join his startup in Palo Alto. 38
- Said I'd absolutely have to talk to you first, 19
- that you are my first priority in life. 7
- But I'm just so confused on so many levels. 15
- Please call me! I love you so much! Alice 0

Continuous martingales

- Cassandra is a rational person. She subjective probability estimates in real time so fast that they can be viewed as continuous martingales.
- She uses the phrase "I think X" in a precise way: it means that $P(X) > 1/2$.
- Cassandra thinks she will win her tennis match today. However, she thinks that she will at some point think she won't win. She does not think that she will ever think that she won't at some point think she will win.
- What's the probability that Cassandra will win? (Give the full range of possibilities.)

Theorems

- L^p **convergence theorem:** If X_n is martingale with $\sup E|X_n|^p < \infty$ where $p > 1$ then $X_n \to X$ a.s. and in L^p.
- **Orthogonal increment theorem:** Let X_n be a martingale with $EX_n^2 < \infty$ for all n. If $m \le n$ and $Y \in \mathcal{F}_m$ with $EY^2 < \infty$, then $E\big((X_n - X_m)Y\big) = 0$.
- **Cond. variance theorem:** If X_n is martingale, $EX_n^2 < \infty$ for all n, then $E\big((X_n - X_m)^2|\mathcal{F}_m\big) = E(X_n^2|\mathcal{F}_m) - X_m^2$.
- **"Accumulated variance" theorems:** Consider martingale X_n with $EX_n^2 < \infty$ for all n. By Doob, can write $X_n^2 = M_n + A_n$ where M_n is a martingale, and

$$A_n = \sum_{m=1}^{n} E(X_m^2|\mathcal{F}_{m-1}) - X_{m-1}^2 = \sum_{m=1}^{n} E\big((X_m - X_{m-1})^2|\mathcal{F}_{m-1}\big).$$

Then $E(\sup_m |X_m|^2) \le 4EA_\infty$. And $\lim_{n\to\infty} X_n$ exists and is finite a.s. on $\{A_\infty < \infty\}$.

Uniform integrability

- Say X_i, $i \in I$, are uniform integrable if
$$\lim_{M\to\infty}\left(\sup_{i\in I} E(|X_i|;|X_i|>M)\right)=0.$$
- Example: Given $(\Omega, \mathcal{F}_0, P)$ and $X \in L^1$, then a uniformly integral family is given by $\{E(X|\mathcal{F})\}$ (where $\mathcal{F}$ ranges over all σ-algebras contained in $\mathcal{F}_0$).
- **Theorem:** If $X_n \to X$ in probability then the following are equivalent:
  - X_n are uniformly integrable
  - $X_n \to X$ in L^1
  - $E|X_n| \to E|X| < \infty$
- **Proof idea:** They all amount to controlling "contribution to expectation from values near infinity".

Submartingale convergence

- **Submartingale convergence theorem:** The following are equivalent for a submartingale:
  - It's uniformly integrable.
  - It converges a.s. and in L^1.
  - It converges in L^1.
- **Proof idea:** First implies second: uniform integrability implies $\sup E|X_n| < \infty$, martingale convergence then implies $X_n \to X$ a.s., and previous result implies $X_n \to X$ in probability. Easier to see second implies third, third implies first.

Martingale convergence

- **Martingale convergence theorem:** The following are equivalent for a martingale:
  - It's uniformly integrable.
  - It converges a.s. and in L^1.
  - It converges in L^1.
  - There is an integrable random variable X so that $X_n = E(X|\mathcal{F}_n)$.
  - In other words, every uniformly integrable martingale can be interpreted as a "revised expectation given latest information" sequence.

Backwards martingales

- Suppose $E(X_{n+1}|\mathcal{F}_n) = X$ with $n \leq 0$ (and $\mathcal{F}_n$ increasing as n increases).
- Kind of like conditional expectation given less and less an information (as $n \to -\infty$)
- **Theorem:** $X_{-\infty} = \lim_{n\to-\infty} X_n$ exists a.s. and in L^1.
- **Proof idea:** Use upcrosing inequality to show expected number of upcrossings of any interval is finite. Since $X_n = E(X_0|\mathcal{F}_n)$ the X_n are uniformly integrable, and we can deduce convergence in L^1.

General optional stopping theorem

- Let X_n be a uniformly integrable submartingale.
- **Theorem:** For any stopping time N, $X_{N \wedge n}$ is uniformly integrable.
- **Theorem:** If $E|X_n| < \infty$ and $X_n 1_{(N>n)}$ is uniformly integrable, then $X_{N \wedge n}$ is uniformly integrable.
- **Theorem:** For any stopping time $N \leq \infty$, we have $EX_0 \leq EX_N \leq EX_\infty$ where $X_\infty = \lim X_n$.
- **Fairly general form of optional stopping theorem:** If $L \leq M$ are stopping times and $Y_{M \wedge n}$ is a uniformly integrable submartingale, then $EY_L \leq EY_M$ and $Y_L \leq E(Y_M | \mathcal{F}_L)$.