18.175: Lecture 26

More on martingales

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Outline

Outline

Conditional expectation

Regular conditional probabilities

Martingales

Arcsin law, other SRW stories

Recall: conditional expectation

- Say we're given a probability space $(\Omega, \mathcal{F}_0, P)$ and a σ-field $\mathcal{F} \subset \mathcal{F}_0$ and a random variable X measurable w.r.t. $\mathcal{F}_0$, with $E|X| < \infty$. The **conditional expectation of X given $\mathcal{F}$** is a new random variable, which we can denote by $Y = E(X|\mathcal{F})$.
- We require that Y is $\mathcal{F}$ measurable and that for all A in $\mathcal{F}$, we have $\int_A X dP = \int_A Y dP$.
- Any Y satisfying these properties is called a **version** of $E(X|\mathcal{F})$.
- **Theorem:** Up to redefinition on a measure zero set, the random variable $E(X|\mathcal{F})$ exists and is unique.
- This follows from Radon-Nikodym theorem.

Conditional expectation observations

- Linearity: $E(aX+Y|\mathcal{F}) = aE(X|\mathcal{F}) + E(Y|\mathcal{F})$.
- If $X \leq Y$ then $E(E|\mathcal{F}) \leq E(Y|\mathcal{F})$.
- If $X_n \geq 0$ and $X_n \uparrow X$ with $EX < \infty$, then $E(X_n|\mathcal{F}) \uparrow E(X|\mathcal{F})$ (by dominated convergence).
- If $\mathcal{F}_1 \subset \mathcal{F}_2$ then
  - $E(E(X|\mathcal{F}_1)|\mathcal{F}_2) = E(X|\mathcal{F}_1)$.
  - $E(E(X|\mathcal{F}_2)|\mathcal{F}_1) = E(X|\mathcal{F}_1)$.
- Second is kind of interesting: says, after I learn $\mathcal{F}_1$, my best guess of what my best guess for X will be after learning $\mathcal{F}_2$ is simply my current best guess for X.
- Deduce that $E(X|\mathcal{F}_i)$ is a martingale if $\mathcal{F}_i$ is an increasing sequence of σ-algebras and $E(|X|) < \infty$.

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Regular conditional probability

- Consider probability space $(\Omega, \mathcal{F}, P)$, a measurable map $X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ and $\mathcal{G} \subset \mathcal{F}$ a σ-field. Then $\mu : \Omega \times \mathcal{S} \to [0, 1]$ is a **regular conditional distribution for X given $\mathcal{G}$** if
  - For each A, $\omega \to \mu(\omega, A)$ is a version of $P(X \in A|\mathcal{G})$.
  - For a.e. ω, $A \to \mu(\omega, A)$ is a probability measure on $(S, \mathcal{S})$.
- **Theorem:** Regular conditional probabilities exist if $(S, \mathcal{S})$ is nice.

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Martingales

- Let $\mathcal{F}_n$ be increasing sequence of σ-fields (called a **filtration**).
- A sequence X_n is **adapted** to $\mathcal{F}_n$ if $X_n \in \mathcal{F}_n$ for all n. If X_n is an adapted sequence (with $E|X_n| < \infty$) then it is called a **martingale** if
$$E(X_{n+1}|\mathcal{F}_n) = X_n$$
for all n. It's a **supermartingale** (resp., **submartingale**) if same thing holds with $=$ replaced by $\leq$ (resp., $\geq$).

Martingale observations

- **Claim:** If X_n is a supermartingale then for $n > m$ we have $E(X_n|\mathcal{F}_m) \leq X_m$.
- **Proof idea:** Follows if $n = m + 1$ by definition; take $n = m + k$ and use induction on k.
- Similar result holds for submartingales. Also, if X_n is a martingale and $n > m$ then $E(X_n|\mathcal{F}_m) = X_m$.
- **Claim:** if X_n is a martingale w.r.t. $\mathcal{F}_n$ and ϕ is convex with $E|\phi(X_n)| < \infty$ then $\phi(X_n)$ is a submartingale.
- **Proof idea:** Immediate from Jensen's inequality and martingale definition.
- Example: take $\phi(x) = \max\{x, 0\}$.

Predictable sequence

- Call H_n **predictable** if each $H+n$ is $\mathcal{F}_{n-1}$ measurable.
- Maybe H_n represents amount of shares of asset investor has at nth stage.
- Write $(H \cdot X)_n = \sum_{m=1}^n H_m(X_m - X_{m-1})$.
- **Observe:** If X_n is a supermartingale and the $H_n \geq 0$ are bounded, then $(H \cdot X)_n$ is a supermartingale.
- Example: take $H_n = 1_{N \geq n}$ for stopping time N.

Two big results

- **Optional stopping theorem:** Can't make money in expectation by timing sale of asset whose price is non-negative martingale.
- **Proof:** Just a special case of statement about $(H \cdot X)$.
- **Martingale convergence:** A non-negative martingale almost surely has a limit.
- **Idea of proof:** Count upcrossings (times martingale crosses a fixed interval) and devise gambling strategy that makes lots of money if the number of these is not a.s. finite.

Problems

- How many primary candidates ever get above twenty percent in expected probability of victory? (Asked by Aldous.)
- Compute probability of having conditional probability reach a before b.

Wald

- **Wald's equation:** Let X_i be i.i.d. with $E|X_i| < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = EX_1 EN$.
- **Wald's second equation:** Let X_i be i.i.d. with $E|X_i| = 0$ and $EX_i^2 = \sigma^2 < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = \sigma^2 EN$.

Wald applications to SRW

- $S_0 = a \in \mathbb{Z}$ and at each time step S_j independently changes by ± 1 according to a fair coin toss. Fix $A \in \mathbb{Z}$ and let $N = \inf\{k : S_k \in \{0, A\}$. What is $\mathbb{E}S_N$?
- What is $\mathbb{E}N$?

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Reflection principle

- How many walks from $(0, x)$ to (n, y) that don't cross the horizontal axis?
- Try counting walks that *do* cross by giving bijection to walks from $(0, -x)$ to (n, y).

Ballot Theorem

- Suppose that in election candidate A gets α votes and B gets $\beta < \alpha$ votes. What's probability that A is ahead throughout the counting?
- Answer: $(\alpha - \beta)/(\alpha + \beta)$. Can be proved using reflection principle.

Arcsin theorem

- Theorem for last hitting time.
- Theorem for amount of positive positive time.

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